Proportionality in Approval-Based Elections With a Variable Number of Winners

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Abstract

We study proportionality in approval-based multiwinner elections with a variable number of winners, where both the size and identity of the winning committee are informed by voters' opinions. While proportionality has been studied in multiwinner elections with a fixed number of winners, it has not been considered in the variable number of winners setting. The measure of proportionality we consider is *average satisfaction (AS)*, which intuitively measures the number of agreements on average between sufficiently large and cohesive groups of voters and the output of the voting rule. First, we show an upper bound on AS that any deterministic rule can provide, and that straightforward adaptations of deterministic rules from the fixed number of winners setting do not achieve better than a $1/2$ approximation to AS even for large numbers of candidates. We then prove that a natural randomized rule achieves a $29/32$ approximation to AS.

1 Introduction

We study *multiwinner approval-based elections*, where a group of agents, or voters, selects a committee from a set of candidates based on the agents' preferences. Each agent expresses her preferences through an approval vote, where she designates a subset of candidates she approves for the committee, and all votes are then aggregated to select a winning committee from the pool of candidates.

Some multiwinner elections include a fixed committee size: the outcome must fill exactly k seats on a committee. This is known as the fixed number of winners (FNW) setting, and there is a large body of work on the complexity and proportionality of various voting rules in the FNW setting [Aziz *et al.*, 2017; Sánchez-Fernández *et al.*, 2017; Aziz *et al.*, 2018; Brill *et al.*, 2017; Peters and Skowron, 2019; Skowron *et al.*, 2017b]. In contrast, we are interested in the setting in which there is no a priori fixed committee size, also known as the variable number of winners (VNW) setting. In this case, both the size of the committee and the candidates chosen to sit on the committee are informed by agents' votes.

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We present a setting where VNW elections are a natural fit; Faliszewski *et al.* [2017] discuss others.

Consider an election that consists of a series of ballot measures, where each ballot question can easily be reversed such that "Yes" becomes "No" and "No" becomes "Yes". This is a practical concern, as ballots are often deliberately constructed such that a "Yes" on one question represents a vote in favor of *upholding* a current statute, while a "Yes" on another question down the ballot represents a vote in favor of *repealing* a current statute [Mueller, 1969]. In this case, voters derive utility from every decision they agree with, whether it is an approval vote or a disapproval vote. Note that, because there is no set number of measures that must be "elected" (i.e., passed), this constitutes a VNW election.

It can be important to ensure that the selected alternatives are chosen in a proportional manner. For instance, in the case of ballot measures, we may want to ensure that all groups in the electorate are satisfied with at least some of the outcomes. In other words, a small majority of the electorate should not be able to overrule a sizable minority on *every* ballot measure.

In order to study proportionality in FNW elections, researchers have proposed the axioms of justified representation (JR), proportional justified representation (PJR), extended justified representation (EJR), and average satisfaction (AS) [Aziz *et al.*, 2017; Sánchez-Fernández *et al.*, 2017], which capture the intuition that all sufficiently large groups that agree on sufficiently many candidates should achieve some measure of satisfaction. However, to our knowledge, we are the first to study representation in VNW elections.

Our Contributions: Our main research goal is to study proportionality in multiwinner elections with a variable number of winners. In particular, we study the proportionality measure of average satisfaction (AS) and show that there is a separation between the performance of deterministic and randomized voting rules.

As our first contribution, we develop a framework for thinking about proportionality in VNW elections. Previous work on proportionality in FNW elections is largely based on the concept of justified representation (and extensions thereof). However, as we discuss in Section 3, JR-based notions of proportionality are less compelling in VNW elections than in FNW elections. Therefore, we instead base our approach on the concept of average satisfaction, which is arguably a more robust version of justified representation.

Second, in Section 4, we consider the proportionality guarantees of deterministic rules in the VNW setting. We extend three existing deterministic rules for the FNW setting to the VNW setting, and show that these rules do not guarantee good approximations to average satisfaction. We also prove upper bounds on the level of average satisfaction that any deterministic rule can provide.

Finally, in Section 5, motivated by the shortcomings of deterministic rules, we turn our attention to randomized rules and show that a natural randomized rule provides a good approximation to average satisfaction.

Related Work

There is a significant body of work studying proportionality in FNW elections. As mentioned above, Aziz *et al.* [2017] put forward the compelling axiom of justified representation (JR), as well as a stronger version of this axiom, extended justified representation (EJR) to capture the notion that any sufficiently large and cohesive group of voters deserves some measure of representation in the elected committee. Sánchez-Fernández *et al.* [2017] build on this idea by introducing the intermediate axiom of proportional justified representation (PJR), a relaxation of EJR that is more stringent than JR.

Average satisfaction (AS) was first defined by Sánchez-Fernández *et al.* [2017], who study the average satisfaction guaranteed by extended justified representation (EJR). Further work by Aziz *et al.* [2018] shows that Proportional Approval Voting (PAV) guarantees a level of average satisfaction that implies EJR. Additionally, Skowron *et al.* [2017a] extend the notion of average satisfaction to the context of complete rankings as opposed to committee selection. Further work by Skowron [2018] studies the proportionality degree of various multiwinner rules by considering the average satisfaction of all groups of a certain size.

There is also a significant body of work studying VNW elections; however, to the best of our knowledge, none of the proposed rules satisfy proportionality (and, in general, that is not their goal). Kilgour [2016] proposes a multitude of rules for VNW elections, including satisfaction approval voting and variants thereof. In a related vein, Kilgour *et al.* [2006] and Brams *et al.* [2007] study the minimax and minisum rules for selecting a committee in the VNW setting. Fishburn and Pekeč [2004] study threshold approaches to committee selection, which are VNW rules in the sense that the size of the selected committee depends on the approval votes. Additionally, the Mean Rule [Duddy *et al.*, 2016] and Borda Mean Rule [Brandl and Peters, 2019] can be seen as VNW rules when given approval votes. Finally, Faliszewski *et al.* [2017] study the computational complexity of various VNW rules, but do not consider proportionality in their analysis.

2 Preliminaries

Let $N = \{v_1, \dots, v_n\}$ be a set of n voters and $C = \{c_1, \dots, c_m\}$ be a set of m candidates. For every voter v_i, denote by $A_i \subseteq C$ the set of candidates that are *approved* by v_i. A preference profile $\mathbf{A} = \{A_1, \dots, A_n\}$ is the set of all voter preferences A_i.

A variable number of winners (VNW) voting rule f takes as input a preference profile $\mathbf{A}$ and outputs some set of candidates $f(\mathbf{A}) \subseteq C$. Note that we allow $f(\mathbf{A}) = \emptyset$ or $f(\mathbf{A}) = C$. We will also consider randomized VNW voting rules that output a distribution over sets of candidates.

Throughout this paper, we will denote by W the set of candidates included in the committee, and we will denote by $C \setminus W$ the set of candidates excluded from the committee.

We say that a group of voters $V \subseteq N$ is ℓ-large if $|V| \geq \ell \cdot \frac{n}{m}$, and ℓ-cohesive if $|\cap_{i \in V} A_i| + |\cap_{i \in V} C \setminus A_i| \geq \ell$. We will also say that a group of voters V *agrees* on a candidate c_j if $c_j \in A_i$ for all $i \in V$ or $c_j \notin A_i$ for all $i \in V$. Otherwise, we say that V *disagrees* on c_j. Intuitively, a group of voters is ℓ-large and ℓ-cohesive if they constitute an ℓ/m fraction of all voters who agree on ℓ out of m candidates.

In our work, we consider a different measure of representation than in the FNW setting. In the FNW setting, voters derive utility from the number of their approved candidates elected to the committee. However, this definition cannot be easily adapted to the VNW setting because then a rule could maximally satisfy all voters by including all candidates on the committee. Therefore, we assume that voters derive utility from agreeing with the placement of candidates either on the committee or not on the committee. For instance, in an election with two candidates, c_1 and c_2, if a voter i has approval set $A_i = \{c_1\}$ (i.e., she approves c_1 and disapproves c_2), then she receives one unit of utility for the output committee $\{c_1, c_2\}$ because she agrees with the inclusion of c_1 but disagrees with the inclusion of c_2.

With this in mind, the following definition of average satisfaction is adapted from the definition of Sánchez-Fernández *et al.* [2017] in the FNW setting.

Definition 1. *Given a set of candidates $W \subseteq C$, the* average satisfaction *of a group of voters $V \subseteq N$ is*

$$avs_W(V) = \frac{1}{|V|} \sum_{i \in V} (|A_i \cap W| + |(C \setminus A_i) \cap (C \setminus W)|).$$

We can now define AS in the VNW setting.[1] The intuition behind the following definition is that any sufficiently large and cohesive group of voters deserves to be adequately represented *on average*, which is a departure from justified representation-based axioms that have been studied in the FNW setting. Intuitively, JR-like notions of proportionality only require that some member of each cohesive group is represented to some extent, whereas average satisfaction requires all members of each cohesive group to be represented (at least on average).

Definition 2. *A set of candidates $W \subseteq C$ satisfies α-AS if, for all ℓ-large and ℓ-cohesive groups of voters $V \subseteq N$, $avs_W(V) \geq \alpha \cdot \ell$ for all $\ell \in [m]$. For brevity, we refer to the special case of 1-AS as AS.*

The following example demonstrates cohesiveness and average satisfaction.

[1]Note that we overload the use of the term "average satisfaction" to refer to both the numerical quantity from Definition 1 (average satisfaction) as well as the axiomatic property in Definition 2 (AS).

Example 1. *Consider the following profile with $n = 8$ voters, $v_1, \ldots, v_8$, and $m = 4$ candidates, $c_1, \ldots, c_4$, with preferences*

$$A_1 = A_2 = \{c_1, c_2, c_3, c_4\} \qquad A_6 = \{c_2, c_3\}$$
$$A_3 = A_4 = \{c_1, c_2\} \qquad A_7 = \{c_3\}$$
$$A_5 = \{c_1, c_3\} \qquad A_8 = \{c_4\}.$$

Now, consider the output $W = \{c_4\}$. Note that each voter agrees with the output on the placement of at least one candidate, so for any 1-large and 1-cohesive group $V_{(1)}$ (i.e., a group of $1 \cdot \frac{n}{m} = 2$ voters who agrees on the placement of 1 candidate), $avs_W(V_{(1)}) \geq 1$. Furthermore, note that there is only one 2-large and 2-cohesive group of voters: v_1, v_2, v_3, and v_4 agree on the placement of c_1 and c_2, but disagree on the placement of c_3 and c_4, so they constitute a 2-large group of voters who agree on 2 candidates. Let $V_{(2)} = \{v_1, v_2, v_3, v_4\}$. Note that $avs_W(V_{(2)}) = 1$ because each $v \in V$ agrees with W on exactly one placement, but because this group of voters is 2-large and 2-cohesive, we see that W only satisfies 1/2-AS in this scenario.

Given our definition of voter satisfaction, we can straightforwardly extend the following deterministic multiwinner rules from the FNW setting to the VNW setting.

Proportional Approval Voting (PAV): Under the PAV rule [Thiele, 1895], voter i derives utility $H_k = 1 + 1/2 + \cdots + 1/k$ from a committee W, where $k = |A_i \cap W| + |(C \setminus A_i) \cap (C \setminus W)|$ is the number of candidate placements that i agrees with. The goal of PAV is to maximize the sum of all voters' utilities, and thus PAV outputs the subset $W \subseteq C$ with highest PAV-score.

Sequential Phragmén (seq-Phragmén): The seq-Phragmén rule [Phragmén, 1899; Janson, 2016; Brill *et al.*, 2017] is defined as follows. Each candidate carries a load of one unit, and this load is distributed among voters who agree with the placement of this candidate in either the included set or excluded set. The seq-Phragmén rule proceeds iteratively by, in each round, placing the candidate that results in the smallest increase in the maximal load of any voter.

Let $x_i^{(t)}$ denote the load of voter i, and $s^{(t)}$ the maximal load, after t candidates have been placed. All voters start out with no load, $x_i^{(0)} = 0$. Furthermore, let $N_j = \{i \in N : c_j \in A_i\}$ represent the set of voters that approve of candidate c_j. The maximal voter load if, on the t^{th} placement, candidate c_j is included in the committee is

$$s^{(t)}(c_j) = \frac{1 + \sum_{i \in N_j} x_i^{(t-1)}}{|N_j|},$$

and the maximal voter load if candidate c_j is excluded from the committee is

$$s^{(t)}(\overline{c_j}) = \frac{1 + \sum_{i \in N \setminus N_j} x_i^{(t-1)}}{|N \setminus N_j|}$$

because the load is distributed so as to equalize the loads of all voters who agree with the placement of c_j. At each step t, seq-Phragmén places the candidate c_j that minimizes $\min(s^{(t)}(c_j), s^{(t)}(\overline{c_j}))$ and updates voter loads accordingly: in the case that c_j is included in the committee,

$$x_i^{(t)} = \begin{cases} s^{(t)}(c_j) & \text{if } i \in N_j \\ x_i^{(t-1)} & \text{otherwise,} \end{cases}$$

and in the case that c_j is excluded from the committee,

$$x_i^{(t)} = \begin{cases} s^{(t)}(\overline{c_j}) & \text{if } i \in N \setminus N_j \\ x_i^{(t-1)} & \text{otherwise.} \end{cases}$$

This rule proceeds until all candidates have been placed, and then returns the included and excluded candidates.

Rule X: Rule X [Peters and Skowron, 2019] allocates each voter a budget of one dollar, which they then spend on placing candidates either in the included set or excluded set. Placing a candidate costs n/m dollars, and the set of voters who agree on the placement of this candidate must be able to collectively afford the placement. The rule starts with an empty included set W and an empty excluded set $\overline{W}$, and it iteratively places candidates in the committee or its complement as follows.

Let $b_i(t)$ be the amount of money that voter i has remaining after the t^{th} candidate is placed; i.e., $b_i(0) = 1$ for all voters $v_i \in N$. At the t^{th} step, we say that a candidate $c \notin W \cup \overline{W}$ is q-affordable for some $q \geq 0$ if

$$\max\left(\sum_{i: c \in A_i} \min(q, b_i(t-1)), \sum_{i: c \in C \setminus A_i} \min(q, b_i(t-1))\right) \geq n/m.$$

In other words, candidate c is q-affordable if it can be placed in either the included or excluded set while voters who approve or disapprove of c each pay a maximum of q dollars. If no candidate is q-affordable for any $q \geq 0$, then the rule stops, placing the current set of included candidates into W, the current set of excluded candidates into $C \setminus W$, and placing arbitrarily any candidates not already put into W or into $C \setminus W$. Else, the rule places the candidate which is q-affordable for the minimum value q in the approved or disapproved committee, according to voter preferences. Each voter who agrees with this placement has their budget updated to $b_i(t) = b_i(t-1) - \min(q, b_i(t-1))$, and the process continues.

3 Justified Representation in VNW Elections

In order to build intuition about why we focus on AS instead of (E/P)JR, we begin by defining JR, PJR, and EJR for VNW elections. In each case, the definition is a straightforward adaptation of the corresponding definition for the FNW setting, where we intuitively replace "agreement with members on the committee" with "agreement on the placement of each candidate." We slightly overload notation—namely, JR, PJR, and E JR—from the FNW setting in the following definitions.

Definition 3 (JR)**.** *Consider a ballot profile* $\mathbf{A}$. *A set of candidates $W \subseteq C$ satisfies* justified representation (JR) *with respect to* $\mathbf{A}$ *if, for all sets of 1-large and 1-cohesive voters N^*, there exists an $i \in N^*$ such that $|A_i \cap W| + |(C \setminus A_i) \cap (C \setminus W)| \geq 1$.*

Definition 4 (PJR). *Consider a ballot profile* $\mathbf{A}$. *A set of candidates* $W \subseteq C$ *satisfies* proportional justified representation (PJR) *with respect to* $\mathbf{A}$ *if, for all* ℓ*-large and* ℓ*-cohesive groups of voters* N^*, $|\bigcup_{i\in N^*} A_i \cap W| + |(\bigcup_{i\in N^*}(C\setminus A_i)) \cap (C\setminus W)| \geq \ell$ *for all* $\ell \in [m]$.

Definition 5 (EJR). *Consider a ballot profile* $\mathbf{A}$. *A set of candidates* $W \subseteq C$ *satisfies* extended justified representation (EJR) *with respect to* $\mathbf{A}$ *if, for all* ℓ*-large and* ℓ*-cohesive groups of voters* N^**, there exists an* $i \in N^*$ *such that* $|A_i \cap W| + |(C\setminus A_i) \cap (C \setminus W)| \geq \ell$ *for all* $\ell \in [m]$.

The following example illustrates these definitions.

Example 2. *Consider the same profile as in Example 1 with* $n = 8$ *voters,* $v_1, \dots, v_8$*, and* $m = 4$ *candidates,* $c_1, \dots, c_4$.

Again, consider the output $W = \{c_4\}$. W *satisfies JR because each voter agrees with the output on the placement of at least one candidate. Furthermore,* W *satisfies PJR because, on the only* 2*-large and* 2*-cohesive group of voters,* $\{v_1, v_2, v_3, v_4\}$*, two of them agree with the placement of* c_3 *and two of them agree with the placement of* c_4*. However,* W *does not satisfy EJR because no voter in the coalition agrees with two placements of* W*—they all agree with exactly one placement.*

We also study the relationship between the extensions of PAV, seq-Phragmén, and Rule X, and different notions of justified representation in the VNW setting.

Proposition 1. *PAV satisfies PJR.*

Proof. Assume that there exists an ℓ-large, ℓ-cohesive coalition, G, that agrees on $\ell' \geq \ell$ candidates C_G, but $|\bigcup_{i\in G} A_i \cap W| + |(\bigcup_{i\in G}(C\setminus A_i)) \cap (C\setminus W)| = k < \ell$. We will show that it is always possible to find a new output W' with a higher PAV score than W.

Let W agree with G on the placement of $\beta < \ell$ of the candidates in C_G. Note that $\beta + (m - \ell') = k$ because every candidate that G disagrees on must be contained in either $\bigcup_{i\in G} A_i \cap W$ or $(\bigcup_{i\in G}(C\setminus A_i)) \cap (C\setminus W)$. We therefore have $\beta + (m-\ell') < \ell$ and therefore $\ell + \ell' > \beta + m$.

In order to argue that there exists a W' with higher PAV score than W, we examine what happens if we change the placement of (a random) one of the $\ell' - \beta$ candidates that G agrees upon, but W does not agree with G on. We will show that, in expectation, changing the placement of one of these candidates increases the PAV score. We let $\Delta_{PAV}(G)$ denote the expected total change in PAV score for voters in G and $\Delta_{PAV}(N\setminus G)$ denote the expected total change in PAV score for voters in $N \setminus G$, and analyze these two quantities separately.

For each voter in G, her PAV score increases by at least $1/(k+1) \geq 1/\ell$ on each of the $\ell' - \beta$ possible changes because each member is at most k-satisfied. Furthermore, there are at least $\ell n/m$ voters in G. Therefore,

$$\Delta_{PAV}(G) \geq \frac{\ell n}{m} \cdot \frac{1}{\ell}.$$

For each voter v_i in $N \setminus G$, let v_i agree with the placement of x_i candidates among the $\ell' - \beta$ candidates that may incur a change, and y_i of the remaining candidates, for a total of $x_i + y_i$ agreements with W. Initially, the PAV score of v_i is $H_{x_i+y_i}$. However, the expected PAV score of each v_i in $N\setminus G$ over all $\ell' - \beta$ possible changes is

$$\frac{x_i H_{x_i+y_i-1} + ((\ell'-\beta) - x_i) H_{x_i+y_i+1}}{\ell' - \beta},$$

so the average change in PAV score for each v_i not in G is

$$\begin{aligned}
&\frac{x_i H_{x_i+y_i-1} + ((\ell'-\beta) - x_i) H_{x_i+y_i+1}}{\ell' - \beta} - H_{x_i+y_i} \\
&= \frac{1}{\ell'-\beta}\Big((\ell'-\beta) H_{x_i+y_i-1} \\
&\quad + (\ell' - \beta - x_i)\left(\frac{1}{x_i+y_i} + \frac{1}{x_i+y_i+1}\right)\Big) \\
&\quad - \frac{\ell'-\beta}{\ell'-\beta} H_{x_i+y_i} \\
&= \frac{1}{\ell'-\beta}\left(\frac{-x_i}{x_i+y_i} + \frac{\ell'-\beta-x_i}{x_i+y_i+1}\right) \\
&> \frac{1}{\ell'-\beta} \cdot \frac{-x_i}{x_i+y_i} \\
&> \frac{-1}{\ell'-\beta}.
\end{aligned}$$

Because there are at most $n - \ell n/m$ voters in $N \setminus G$, we have

$$\Delta_{PAV}(N\setminus G) \geq \left(n - \frac{\ell n}{m}\right)\left(-\frac{1}{\ell'-\beta}\right).$$

Therefore, the expected total change in PAV score over all $\ell' - \beta$ potential changes is

$$\begin{aligned}
&\Delta_{PAV}(G) + \Delta_{PAV}(N\setminus G) \\
&\qquad \geq \frac{\ell n}{m} \cdot \frac{1}{\ell} + \left(n - \frac{\ell n}{m}\right)\left(-\frac{1}{\ell'-\beta}\right) \\
&\qquad = \frac{n}{m} - \frac{nm - \ell n}{m(\ell'-\beta)} \\
&\qquad = \frac{n\ell' - n\beta}{m(\ell'-\beta)} - \frac{nm-\ell n}{m(\ell'-\beta)} \\
&\qquad = \frac{n(\ell + \ell' - \beta - m)}{m(\ell'-\beta)} \\
&\qquad > 0,
\end{aligned}$$

where the last step follows because $\ell + \ell' > \beta + m$ and $\ell' > \beta$. Therefore, we have shown the existence of W' with higher PAV score, which is a contradiction. □

Proposition 2. *Seq-Phragmén satisfies PJR.*

Proof. Denote by $C^{(t)}$ the set of candidates placed after t rounds of the seq-Phragmén algorithm, and $W^{(t)} \subseteq C^{(t)}$ the set of candidates included in the committee in the first t rounds. Consider any ℓ-large, ℓ-cohesive group G. We will show that $|\bigcup_{i\in G} A_i \cap W^{(m-k)}| + |(\bigcup_{i\in G}(C^{(m-k)} \setminus A_i)) \cap (C^{(m-k)} \setminus W^{(m-k)})| \geq \ell - k$ for all $0 \leq k \leq m$. In particular, for $k = 0$ this implies that seq-Phragmén satisfies PJR.

We proceed by induction. As our base case, for $k \geq \ell$, it is vacuously true that $|\bigcup_{i\in G} A_i \cap W^{(m-k)}| + |(\bigcup_{i\in G}(C^{(m-k)} \setminus A_i)) \cap (C^{(m-k)} \setminus W^{(m-k)})| \geq \ell - k \geq 0$.

For the induction step, assume that $|\bigcup_{i\in G} A_i \cap W^{(m-k-1)}| + |(\bigcup_{i\in G}(C^{(m-k-1)} \setminus A_i)) \cap (C^{(m-k-1)} \setminus W^{(m-k-1)})| \geq \ell - k - 1$, for some $k < \ell$. Call the candidate that is placed on the $(m-k)^{th}$ step c. If G disagrees on c, then no matter how c is placed, $|\bigcup_{i\in G} A_i \cap W^{(m-k)}| + |(\bigcup_{i\in G}(C^{(m-k)} \setminus A_i)) \cap (C^{(m-k)} \setminus W^{(m-k)})| \geq \ell - k$, We now consider the case where G agrees on c. WLOG, assume that every member in G approves c.

If $|\bigcup_{i\in G} A_i \cap W^{(m-k-1)}| + |(\bigcup_{i\in G}(C^{(m-k-1)} \setminus A_i)) \cap (C^{(m-k-1)} \setminus W^{(m-k-1)})| \geq \ell - k$, then G derives the necessary level of satisfaction regardless of the algorithm's action at round $m-k$. Suppose therefore that $|\bigcup_{i\in G} A_i \cap W^{(m-k-1)}| + |(\bigcup_{i\in G}(C^{(m-k-1)} \setminus A_i)) \cap (C^{(m-k-1)} \setminus W^{(m-k-1)})| = \ell - k - 1$, which implies that the total load on G is at most $\ell - k - 1$. That is, $\sum_{i\in G} x_i^{(m-k-1)} \leq \ell - k - 1$. This means that $N \setminus G$, which is of size at most $n - \ell n/m$ voters, has total load at least $(m-k-1)-(\ell-k-1) = m-\ell$. That is, $\sum_{i\in N\setminus G} x_i^{(m-k-1)} \geq m - \ell$. Therefore, there exists a voter in $N \setminus G$ with load $x_i^{(m-k-1)} \geq \frac{m-\ell}{n-\ell n/m} = \frac{m}{n}$. Suppose that at round $m-k$, a candidate from C_G is excluded from the committee (it is easy to see that all other choices result in the necessary level of satisfaction for G). This implies in particular that the average load among voters in $N \setminus G$ after $m-k$ rounds is strictly greater than m/n. We show that this yields a contradiction.

Return to the first round t at which any voter's load exceeded m/n. That is, $s^{(t)}(c^{(t)}) > m/n$, where $c^{(t)}$ denotes the candidate placed at round t. Now, replace the algorithm's action at round t by including c instead. We can upper bound $s^{(t)}(c)$ by imagining that the entire load from candidate c is incurred by members of G. After including c, the total load on G is $\sum_{i\in G} x_i^{(t)} \leq \ell - k$, and therefore the average voter load among members of G at this point in time is at most

$$\frac{\ell - k}{\ell n/m} \leq \frac{\ell}{\ell n/m} = \frac{m}{n}.$$

Therefore, $s^{(t)}(c) \leq \frac{n}{m} < s^{(t)}(c^{(t)})$, a contradiction. ☐

Proposition 3. *Rule X satisfies PJR but not EJR.*

Proof. We first prove that Rule X satisfies PJR, and then we show that Rule X does not satisfy EJR.

Rule X satisfies PJR: Denote by $C^{(t)}$ the set of candidates placed after t rounds of Rule X, and $W^{(t)} \subseteq C^{(t)}$ the set of candidates included in the committee in the first t rounds. Consider any ℓ-large and ℓ-cohesive group G of size $\ell n/m$. Let G agree on $\ell' \geq \ell$ candidates C_G. WLOG, let G approve all candidates in C_G. We show that if $|\bigcup_{i\in G} A_i \cap W^{(t)}| + |(\bigcup_{i\in G}(C^{(t)} \setminus A_i)) \cap (C^{(t)} \setminus W^{(t)})| < \ell$ then there exists a $(\frac{1}{\ell})$-affordable candidate.

We distinguish two cases: either there still exists an unplaced candidate in C_G, or all of C_G has been placed.

In the first case, we note that no voter $v_i \in G$ has spent more than $\frac{1}{\ell}$ on any single candidate over the first t rounds. If they had, then we can return to the first such round, r, and note that an unplaced candidate $c \in C_G$ can be included at a cost at most $\frac{1}{\ell}$ to each voter who approves c. The cost is at most $\frac{1}{\ell}$ because there are at least $\ell n/m$ voters in G who have each paid at most $\frac{1}{\ell}$ in each of (at most) $\ell - 1$ earlier rounds. Therefore, returning to round $t+1$, we know that each voter in G agrees with at most $\ell - 1$ of the placements over the first t rounds, and paid at most $\frac{1}{\ell}$ for each of these placements, so each of them has at least $\frac{1}{\ell}$ of their budget remaining. Therefore, an unplaced candidate from C_G can be included for a cost of at most $q = \frac{1}{\ell}$ to every voter who agrees with it.

In the second case, we examine how many candidates in C_G are excluded from $W^{(t)}$. All of G approves all of C_G, so the entire cost of excluding any candidate in C_G is borne by $N \setminus G$, which consists of at most $n - \frac{\ell n}{m}$ voters. Since each candidate costs n/m to place, these voters can afford to exclude at most $(n - \ell\frac{n}{m})/(\frac{n}{m}) = m - \ell$ of them. Therefore, $\ell' - (m - \ell)$ of the candidates in C_G must be included in W. This is enough to guarantee PJR, since all of the $m - \ell'$ candidates in $C \setminus C_G$ will be contained in either $\bigcup_{i\in G} A_i \cap W$ or $(\bigcup_{i\in G}(C \setminus A_i)) \cap (C \setminus W)$ regardless of their eventual placement.

Rule X does not satisfy EJR: Consider the following profile with $n = 12$ and $m = 6$ with the following preferences.

$$\begin{aligned} A_1 = \ldots = A_4 &= \{c_1, c_2, c_5\} \\ A_5 = \ldots = A_8 &= \{c_1, c_2, c_6\} \\ A_9 = A_{10} &= \{c_3, c_4, c_5\} \\ A_{11} = A_{12} &= \{c_3, c_4, c_6\}. \end{aligned}$$

The first four actions of Rule X will be to include c_1 and c_2 and exclude c_3 and c_4. After these placements, voters $v_1, \ldots, v_8$ will have no money left, and voters $v_9, \ldots, v_{12}$ will have their entire budgets left. Now, Rule X will perfectly spend the money of $v_9, \ldots, v_{12}$ over candidates c_5 and c_6 (either including both or neither) and the coalition of voters $v_9, \ldots, v_{12}$ will not satisfy EJR because each voter in this coalition is represented exactly once, even though someone in this coalition deserves to be represented twice. ☐

Notably, in the VNW setting, JR and PJR are less compelling notions of representation than in the FNW setting. In particular, whenever an ℓ-cohesive group of voters does not agree on the placement of a particular candidate, PJR automatically counts that candidate toward the group's representation quota, since at least one member of the group agrees with the candidate's placement. In other words, any disagreement within an ℓ-cohesive group results in partial representation, no matter the outcome of the election. This is particularly problematic for JR: any 1-large, 1-cohesive group of voters that disagrees on even a single candidate will never be witness to a violation of JR.

Proposition 3 is also notable because Rule X satisfies EJR for FNW elections, but the straightforward extension of this rule does not satisfy EJR for VNW elections, demonstrating a qualitative difference between proportionality properties in the FNW and VNW settings. It is still an open question whether or not PAV and seq-Phragmén satisfy EJR for VNW elections.

4 Deterministic Rules

We begin by showing an upper bound on the level of average satisfaction that deterministic rules can provide.

Theorem 1. *No deterministic rule satisfies* $(\frac{m-1}{m}+\epsilon)$*-AS for any* m *and any* $\epsilon > 0$*.*

Proof. First, suppose that m is odd. Then set $n = 2$, with $A_1 = \{c_1, \dots, c_m\}$ and $A_2 = \emptyset$. Without loss of generality, suppose that the output W is such that $|W| > \frac{m}{2}$. But then voter v_2 is an $\frac{m}{2}$-large, $\frac{m}{2}$-cohesive group with average satisfaction at most $\frac{m-1}{2}$, which yields an $(\frac{m-1}{m})$-AS approximation.

Next, suppose m is even, and set $n = 4m$. Consider the profile

$$A_1 = \{c_1, \dots, c_m\} \qquad A_3 = \{c_m\}$$
$$A_2 = \{c_1, \dots, c_{m-1}\} \qquad A_4 = \emptyset$$

Again, without loss of generality, suppose that the output W is such that $|W| \geq \frac{m}{2}$. We consider two cases. In the first case, suppose that the output W has $|W| \geq \frac{m}{2} + 1$. Consider the $\frac{m}{2}$-large, $\frac{m}{2}$-cohesive group of voters $V = \{v_3, v_4\}$. We have

$$avs_W(V) \leq \frac{1}{2}(m - |W| + m - |W| + 1) \leq \frac{m-1}{2}$$

which yields at most an $(\frac{m-1}{m})$-AS approximation.

In the second case, suppose that the output W has $|W| = \frac{m}{2}$. Suppose that $c_m \notin W$ (the case of $c_m \in W$ follows symmetrically). Then again consider $V = \{v_3, v_4\}$. We have

$$avs_W(V) \leq \frac{1}{2}\left(\frac{m}{2} - 1 + \frac{m}{2}\right) = \frac{m-1}{2}$$

again yielding an $(\frac{m-1}{m})$-AS approximation. This completes the proof. ☐

Theorem 1 leaves open the possibility that there exists a deterministic rule that provides quite good average satisfaction guarantees when the number of candidates is large. Finding such a rule or lowering the upper bound is an interesting open question. However, we show that none of the natural adaptations of FNW rules that we consider is able to guarantee better than a 0.5 approximation to AS even when m is large.

Theorem 2. *PAV does not satisfy* $(0.5+\epsilon)$*-AS, for any* $\epsilon > 0$ *for* $m \geq 2$*.*

Proof. Consider a profile with $n = 2m$ voters with preferences

$$A_1 = \ldots = A_{m-1} = \{c_1, \dots, c_m\}$$
$$A_m = \ldots = A_{2m-2} = \{c_1, \dots, c_{m-1}\}$$
$$A_{2m-1} = \{c_m\}$$
$$A_{2m} = \emptyset.$$

This profile is symmetric in c_m, so without loss of generality suppose that c_m is included. Suppose that some $k - 1 < m - 1$ of the candidates $c_1, \dots, c_{m-1}$ are included. Then, the *change* in PAV score that would result from including an additional candidate is

$$\frac{m-1}{k+1} + \frac{m-1}{k} - \frac{1}{m-k} - \frac{1}{m-k+1}$$
$$\geq \frac{m-1}{m} + 1 - 1 - \frac{1}{2} \geq 0,$$

where the first inequality holds because $k < m$.

Therefore, the maximum PAV score is achieved when all candidates $c_1, \dots, c_{m-1}$ are included. But then the group $N^* = \{v_{2m-1}, v_{2m}\}$ is 1-large and 1-cohesive but is only satisfied 0.5 times on average. ☐

Theorem 3. *seq-Phragmén does not satisfy* $(0.5+\epsilon)$*-AS, for any* $\epsilon > 0$ *for* $m \geq 2$*.*

Proof. Consider the same profile as in the proof of Theorem 2. It is easy to check that seq-Phragmén begins by including candidates $c_1, \dots, c_{m-2}$, after which each voter of the first and second type has load $\frac{m-2}{2(m-1)}$. In the $(m-1)$-th round, the algorithm has four choices: to include or exclude c_{m-1}, or to include or exclude c_m.

Including c_{m-1} results in a load of $\frac{m-1}{2(m-1)} = \frac{1}{2}$ on voters $v_1, \dots, v_{2m-2}$. Excluding c_{m-1} results in a load of $\frac{1}{2}$ to voters v_{2m-1} and v_{2m}. Including c_m (which is symmetric to excluding c_m) results in a load x to voters $v_1, \dots, v_{m-1}, v_{2m-1}$, where x is the solution to $mx - (m-1)\frac{m-2}{2(m-1)} = 1$, which yields a solution of $x = \frac{1}{2}$.

The algorithm is therefore indifferent between all possible actions; breaking ties adversarially yields the inclusion of c_{m-1}. Regardless of the inclusion or exclusion of candidate c_m, the group $N^* = \{v_{2m-1}, v_{2m}\}$ is 1-large and 1-cohesive but is only satisfied 0.5 times on average. ☐

We note that the dependence on tiebreaking in the proof of Theorem 3 can be removed by taking multiple copies of the profile used in the proof and changing the preference of a single voter.

Theorem 4. *Rule X does not satisfy* $(0.5 + \epsilon)$*-AS, for any* $\epsilon > 0$ *for* $m \geq 3$*.*[2]

Proof. Consider the same profile used in the proof of Theorem 2. Rule X begins by including each of candidates $c_1, \dots, c_{m-1}$. Each of these candidates costs $\frac{n}{m(2m-2)} = \frac{1}{m-1}$ for each voter $v_1, \dots, v_{2m-2}$. In comparison, placing the last candidate at any point costs $n/2$ voters $\frac{n/m}{n/2} = \frac{2}{m}$, which is a greater cost than $\frac{1}{m-1}$ when $m \geq 3$. Including each of $c_1, \dots, c_{m-1}$ therefore costs $v_1, \dots, v_{2m-2}$ one dollar each. Regardless of the placement of c_m, the 1-large and 1-cohesive group of voters $N^* = \{v_{2m-1}, v_{2m}\}$ is satisfied only 0.5 times on average. ☐

[2]When $m = 2$, we know from Theorem 1 that no deterministic rule, including Rule X, can achieve better than a 0.5 approximation.

5 Randomized Rules

We now turn our attention to randomized rules in order to achieve better average satisfaction guarantees. A randomized rule is one that outputs a distribution over committees rather than a single committee, and our approximation guarantee will hold in expectation over the possible committees.[3] We consider a simple and natural randomized rule that, for each candidate c_j, includes c_j in the set of winners W with probability equal to the fraction[4] of the voters who approve c_j.

Definition 6. *Given a preference profile* $\mathbf{A}$*, the* Proportional Random Rule (PRR) *independently includes each* $c_j \in C$ *in the winning committee* W *with probability*

$$p_j = \frac{|\{v_i \in N \text{ s.t. } c_j \in A_i\}|}{n}.$$

Theorem 5. *PRR satisfies* $29/32$*-AS in expectation for any* m*.*

In the proof of Theorem 5, it will be helpful to think about the effect that an individual candidate has on the satisfaction of a group G. For an outcome W, a group of voters G, and a candidate c_j, we say that the *contribution from* c_j *to the average satisfaction of* G is $avs_{c_j}(G) = |\{i : c_j \in A_i\}|/|G|$ if $c_j \in W$ or $avs_{c_j}(G) = |\{i : c_j \notin A_i\}|/|G|$ if $c_j \notin W$. Note that $avs_W(G) = \sum_{j=1}^m avs_{c_j}(G)$.

Proof. We prove the result in two steps. First, we show that when $\ell \leq m/3$, PRR achieves an average satisfaction of ℓ; second, we show that when $\ell > m/3$, PRR achieves an average satisfaction of $(29/32)\ell$.

Case 1: $\ell \leq m/3$**.** Consider an ℓ-cohesive group, G, of size $\ell n/m$, and a candidate c_j. Note that it is sufficient to consider groups of size exactly $\ell n/m$ because if there exists an ℓ-cohesive larger group that violates the desired guarantee, there must exist a subset of size $\ell n/m$ that also violates the guarantee. Let $k_A = |\{v_i \in G : c_j \in A_i\}|$ denote the number of voters in G who approve c_j, and $k_D = \ell n/m - k_A$ denote the number of voters in G who disapprove c_j. Without loss of generality, let $k_A \leq k_D$. Further, suppose that x of the voters in $N \setminus G$ approve c_j and $y = n - \ell n/m - x$ voters in $N \setminus G$ disapprove c_j.

The expected contribution from c_j to the average satisfaction of G is

$$\mathbb{E}[avs_{c_j}(G)] = \frac{k_A}{|G|}\left(\frac{k_A + x}{n}\right) + \frac{k_D}{|G|}\left(\frac{k_D + y}{n}\right).$$

Because $k_A \leq k_D$ and $x + y$ is fixed, this expression is minimized when $y = 0$. We therefore have

$$\begin{aligned}\mathbb{E}[avs_{c_j}(G)] &\geq \frac{k_A}{|G|}\left(\frac{k_A + n - \ell n/m}{n}\right) + \frac{k_D}{|G|}\left(\frac{k_D}{n}\right)\\ &= \frac{1}{n|G|}\left(|G|^2 + k_A(n - \ell n/m - 2k_D)\right)\\ &\geq \frac{|G|}{n} = \frac{\ell}{m},\end{aligned}$$

where the inequality holds because $k_D \leq \ell n/m$ by definition, and we can assume $m \geq 3$ because ℓ must be at least 1. Summed over all candidates, the average satisfaction of G is at least ℓ, as required.

Case 2: $\ell > m/3$**.** Consider an ℓ-cohesive group, G, of size $\ell n/m$, and a candidate c_j. Let $k_A = |\{v_i \in G : c_j \in A_i\}|$ denote the number of voters in G who approve c_j, and $k_D = \ell n/m - k_A$ denote the number of voters in G who disapprove c_j. Without loss of generality, let $k_A \leq k_D$. As in the previous case, it is easy to show that the expected contribution from c_j to G's average satisfaction is minimized when all voters in $N \setminus G$ approve c_j.

We therefore have that

$$\mathbb{E}[avs_{c_j}(G)] = \frac{k_A}{|G|}\left(\frac{k_A + n - \ell n/m}{n}\right) + \frac{k_D}{|G|}\left(\frac{k_D}{n}\right).$$

Substituting $k_D = \ell n/m - k_A$, taking the derivative with respect to k_A, and setting to 0 yields

$$\frac{1}{n}\left(4k_A - 3(\ell n/m) + n\right) = 0 \implies k_A = \frac{3\ell n/m - n}{4} > 0,$$

where the inequality follows from the assumption that $\ell > m/3$. Furthermore, the second derivative with respect to k_A is $4/n > 0$, and therefore $k_A = (3\ell n/m - n)/4$ is a local minimum.

The expected contribution from c_j to G's average satisfaction can therefore be as low as

$$\begin{aligned}\mathbb{E}[avs_{c_j}(G)] &= \frac{k_A}{|G|}\left(\frac{k_A + n - \ell n/m}{n}\right) + \frac{k_D}{|G|}\left(\frac{k_D}{n}\right)\\ &= \frac{-\ell}{8m} + \frac{3}{4} - \frac{m}{8\ell}.\end{aligned}$$

We also note that, because G is ℓ-cohesive, there exist at least ℓ candidates that G agrees on. Each of these candidates has

$$avs_{c_j}(G) \geq |G|/n \geq \ell/m,$$

where the first inequality follows from G being ℓ-cohesive and the second from G being ℓ-large.

Summing over the contributions of all candidates, the average satisfaction of G is at least

$$\begin{aligned}&\ell\frac{\ell}{m} + (m - \ell)\left(\frac{3}{4} - \frac{\ell}{8m} - \frac{m}{8\ell}\right)\\ &= \left(\frac{9\ell}{8m} - \frac{m^2}{8\ell^2} - \frac{7}{8} + \frac{7m}{8\ell}\right)\ell. \qquad (1)\end{aligned}$$

Our goal is to lower bound the term in parentheses by $\frac{29}{32}$, thus providing the desired approximation guarantee. Setting

[3] Recent work by Cheng *et al.* [2019] has applied randomization to proportionality in the FNW setting as well.

[4] The marginal probabilities for each candidate being included in the committee are the same under this rule as the random dictatorship rule. The distribution over committees induced by the two rules is different, however.

$\ell = \alpha m$, where $\alpha \in (\frac{1}{3}, 1)$, and differentiating with respect to α yields

$$\frac{d}{d\alpha}\left(\frac{9\alpha}{8} - \frac{1}{8\alpha^2} - \frac{7}{8} + \frac{7}{8\alpha}\right) = \frac{9}{8} + \frac{2}{8\alpha^3} - \frac{7}{8\alpha^2}.$$

Setting equal to 0 yields

$$9\alpha^3 - 7\alpha + 2 = (1+\alpha)(3\alpha-2)(3\alpha-1) = 0,$$

so the only critical point in the interval $\alpha \in (1/3, 1]$ is $\alpha = 2/3$. It is easy to check that the second derivative is positive at $\alpha = 2/3$, so average satisfaction is minimized at this point. Plugging $\ell = 2m/3$ into Equation 1 yields a 29/32 approximation to AS, as desired. □

Guided by Theorem 5, we show that the bound is tight.

Theorem 6. *PRR does not satisfy* $(29/32 + \epsilon)$*-AS for any* $\epsilon > 0$.

Proof. Let $m = 3$ and $n = 12$. Consider the profile

$$A_1 = A_2 = A_3 = A_4 = A_5 = \{c_1, c_2, c_3\}$$
$$A_6 = A_7 = A_8 = \{c_1, c_2\}$$
$$A_9 = A_{10} = A_{11} = A_{12} = \emptyset.$$

In particular, note that the first 8 voters form a 2-large and 2-cohesive group. Then the expected satisfaction of the first five voters is $\frac{2}{3} + \frac{2}{3} + \frac{5}{12} = \frac{21}{12}$ and the expected satisfaction of the next three voters is $\frac{2}{3} + \frac{2}{3} + \frac{7}{12} = \frac{23}{12}$. Taking the average yields $\frac{1}{8}(5\frac{21}{12} + 3\frac{23}{12}) = \frac{29}{16} = \frac{29}{32}\ell$ for $\ell = 2$. □

Whether there exists a randomized rule that achieves better than a 29/32-AS approximation remains an open problem.

Before concluding this section, we note a final interesting and desirable property of PRR: strategyproofness. Since decisions are made on each candidate independently, voters maximize their expected satisfaction by reporting their true approval preferences.

6 Conclusion

We have initiated the study of representation in approval elections with a variable number of winners. We believe that this topic, and the study of VNW elections more generally, deserves further research.

Many open problems remain. In particular, we do not have matching upper and lower bounds for the average satisfaction guarantees that can be provided by deterministic and randomized rules. Determining the existence of rules that satisfy EJR is also an interesting question; while we have argued that natural extensions of JR and PJR make less sense for VNW elections than for FNW, EJR remains a compelling property.

More broadly, we have assumed that voters gain utility whenever they agree with the placement of a candidate, either included or excluded. This is a natural model when the notions of inclusion and exclusion are symmetric, as in the ballot measure example. In other settings it makes sense to consider other utility models. For instance, a natural extension of our model would consider voters who derive different levels of utility for an approved candidate being selected and a disapproved candidate being excluded, or even negative utility from an approved candidate not being selected or a disapproved candidate being included. The latter utility model is reminiscent of rules such as net satisfaction approval voting (NSAV) [Kilgour and Marshall, 2012], and precision and recall metrics in information retrieval. Extending our results to this setting appears nontrivial.

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