# Recent Advances in Fair Resource Allocation

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# Disclaimer

- In this tutorial, we will <u>NOT</u>
  - Assume any prior knowledge of fair division
  - Walk you through tedious, detailed proofs
  - Claim to present a *complete* overview of the entire fair division realm
  - Present (recent) unpublished results
- Instead, we will
  - Focus mostly on the case of "additive preferences" for coherence
    - With some results for and pointers to domains with non-additive preferences
- If you spot any errors, missing results, or incorrect attributions:
  - Please email nisarg@cs.toronto.edu or Rupert.Freeman@microsoft.com

# Outline

- Fairness Axioms
  - Proportionality
  - Envy-freeness
  - Maximin share guarantee
  - Groupwise fairness
    - Core
    - Group envy-freeness
    - Groupwise MMS
    - Group fairness
- Implications of fairness
  - Price of fairness
  - Interplay with strategyproofness and Pareto optimality
  - Restricted cases
- Settings
  - Cake-cutting
  - Homogeneous divisible goods
  - Indivisible goods

# A Generic Resource Allocation Framework

- A set of agents $N = \{1,2, \ldots, n\}$
- A set of resources $M$
  - May be finite or infinite
- Valuations
  - Valuation of agent $i$ is $v_i : 2^M \to \mathbb{R}$
  - Range is $\mathbb{R}_+$ when resources are *goods*, and $\mathbb{R}_-$ when they are *bads*
- Allocations
  - $A = (A_1, \ldots, A_n) \in \Pi_n(M)$ is a partition of resources among agents
    - $A_i \cap A_j = \emptyset, \forall i, j \in N$ and $\cup_{i \in N} A_i = M$
  - A partial allocation $A$ may have $\cup_{i \in N} A_i \neq M$

# Cake Cutting

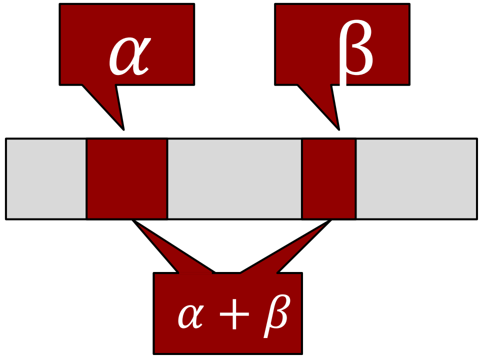
- Formally introduced by Steinhaus [1948]
- Agents: $N = \{1,2,\ldots,n\}$
- Resource (cake): $M = [0,1]$
- Constraints on an allocation $A$
  - The entire cake is allocated (full allocation)
  - Each $A_i \in \mathcal{A}$, where $\mathcal{A}$ is the set of finite unions of disjoint intervals
- Simple allocations
  - Each agent is allocated a single interval
  - Cuts cake at $n-1$ points

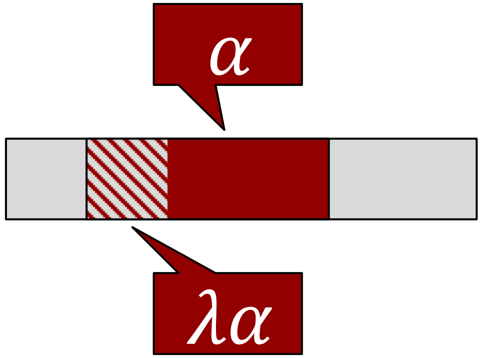
# Agent Valuations

- Each agent $i$ has an integrable density function $f_i : [0,1] \to \mathbb{R}_+$

- For each $X \in \mathcal{A}$, $v_i(X) = \int_{x \in X} f_i(x)dx$

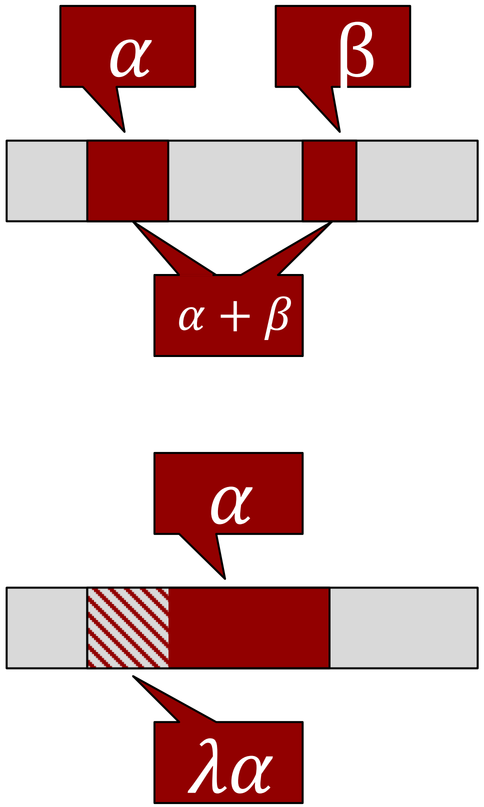
- For normalization, we require $\int_0^1 f_i(x)dx = 1$
  - Without loss of generality



# Agent Valuations

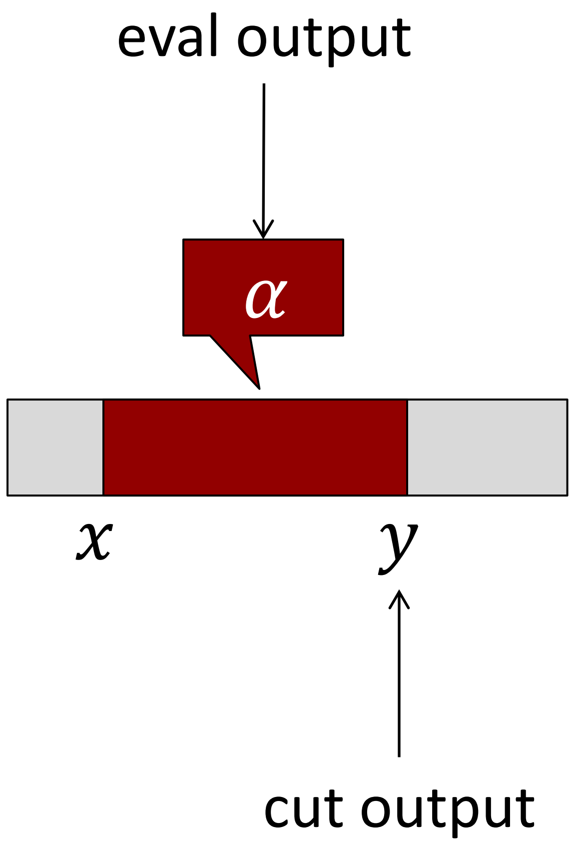
- In this model, the valuations satisfy the following properties
- Normalized: $v_i([0,1]) = 1$
- Divisible: $\forall \lambda \in [0,1]$ and $I = [x, y]$, $\exists z \in [x, y]$ s.t. $v_i([x, z]) = \lambda v_i([x, y])$
- Additive: For disjoint intervals $I$ and $I'$, $v_i(I) + v_i(I') = v_i(I \cup I')$



# Complexity

- Inputs are functions
  - Infinitely many bits may be needed to fully represent the input
  - Query complexity is more useful

- Robertson-Webb Model
  - $\text{Eval}_i(x, y)$ returns $v_i([x, y])$
  - $\text{Cut}_i(x, \alpha)$ returns $y$ such that $v_i([x, y]) = \alpha$



# Three Classic Fairness Desiderata

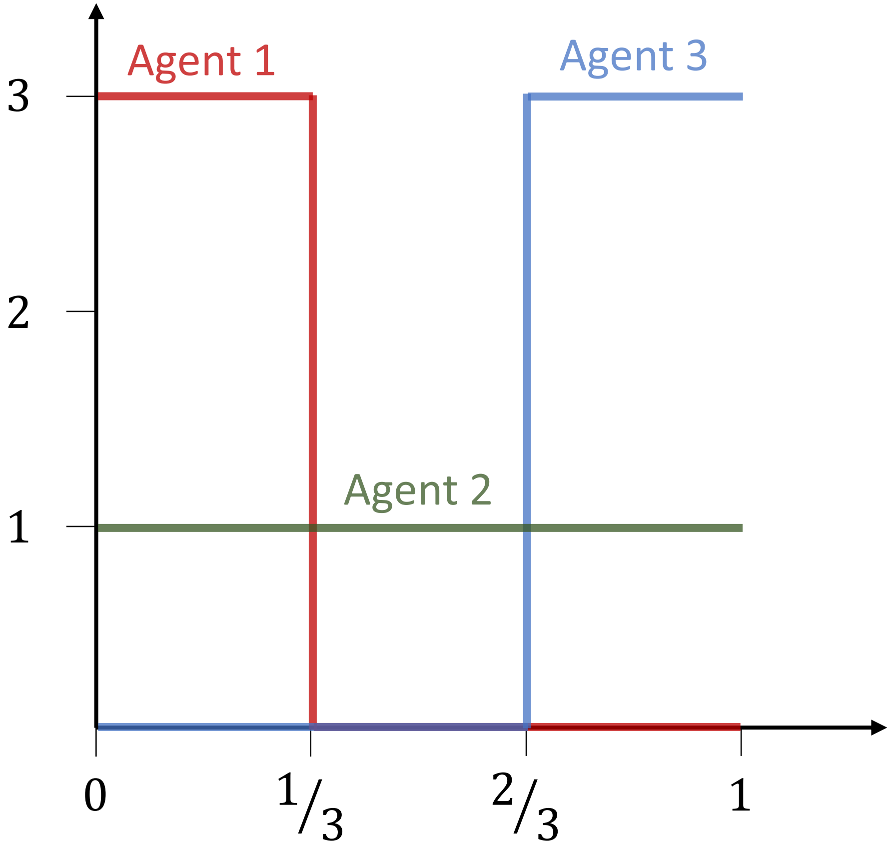
- Proportionality (Prop): $\forall i \in N\colon v_i(A_i) \geq {}^1\!/_n$
  - Each agent should receive her "fair share" of the utility.

- Envy-Freeness (EF): $\forall i, j \in N\colon v_i(A_i) \geq v_i(A_j)$
  - No agent should wish to swap her allocation with another agent.

- Equitability (EQ): $\forall i, j \in N : v_i(A_i) = v_j(A_j)$
  - All agents should have the exact same value for their allocations.
  - No agent should be jealous of what another agent received.

# Example

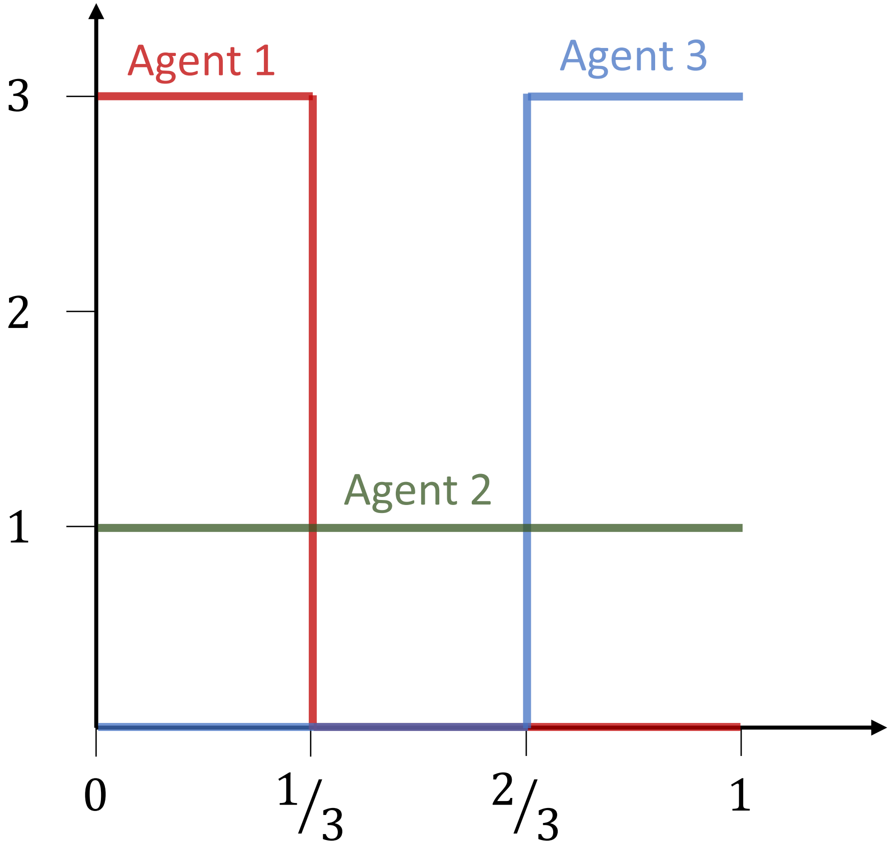
- Value density functions




- Agent 1 wants $[0, {}^1/_3]$ uniformly and does not want anything else
- Agent 2 wants the entire cake uniformly
- Agent 3 wants $[{}^2/_3\,, 1]$ uniformly and does not want anything else

# Example

- Value density functions



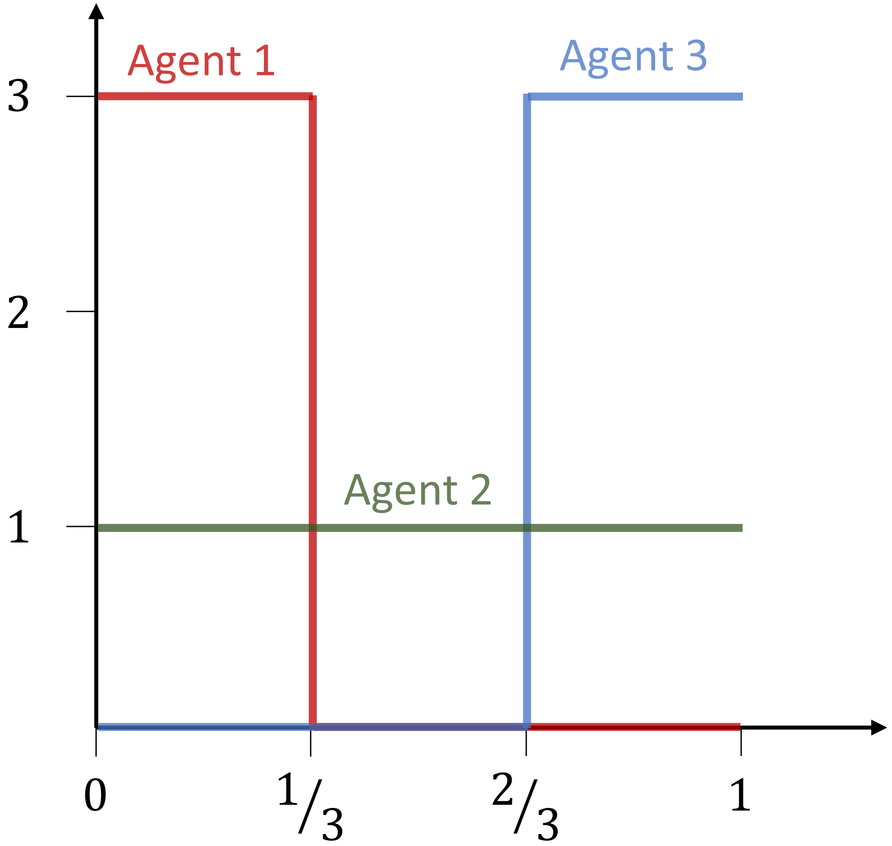

- Consider the following allocation

- $A_1 = [0, {}^1/_9] \Rightarrow v_1(A_1) = {}^1/_3$
- $A_2 = [{}^1/_9, {}^8/_9] \Rightarrow v_2(A_2) = {}^7/_9$
- $A_3 = [{}^8/_9, 1] \Rightarrow v_3(A_3) = {}^1/_3$

- The allocation is proportional, but not envy-free or equitable

# Example

- Value density functions



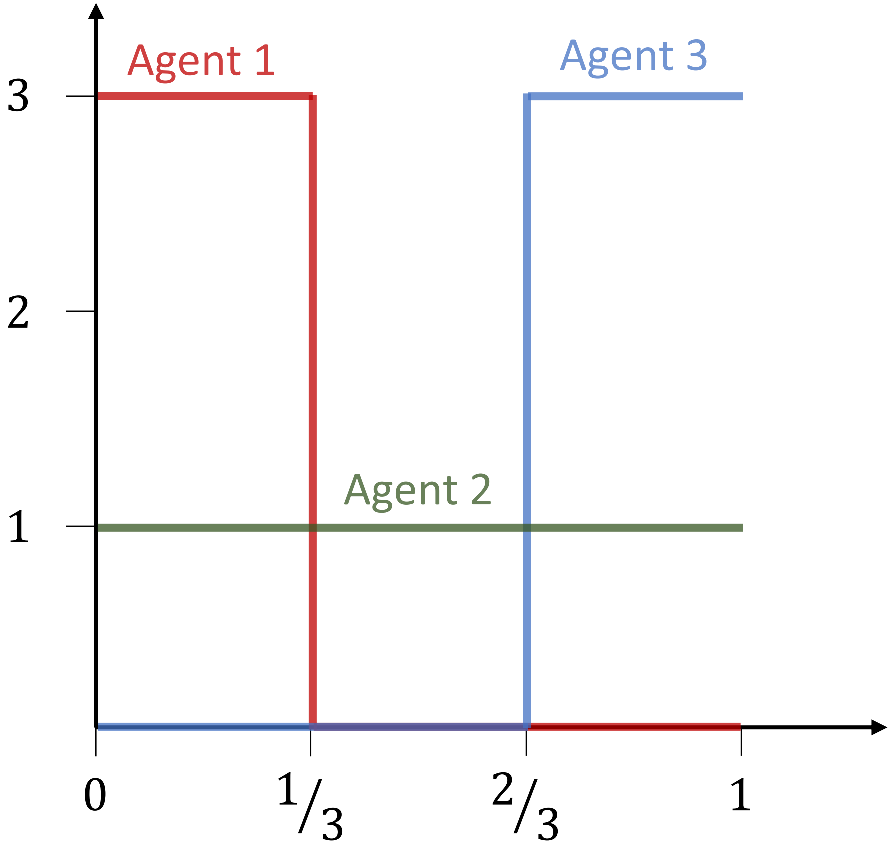

- Consider the following allocation

- $A_1 = [0, {}^1/_6] \Rightarrow v_1(A_1) = {}^1/_2$
- $A_2 = [{}^1/_6, {}^5/_6] \Rightarrow v_2(A_2) = {}^2/_3$
- $A_3 = [{}^5/_6, 1] \Rightarrow v_3(A_3) = {}^1/_2$

- The allocation is proportional and envy-free, but not equitable

# Example

- Value density functions




- Consider the following allocation

- $A_1 = [0, {}^1\!/_5] \Rightarrow v_1(A_1) = {}^3\!/_5$
- $A_2 = [{}^1\!/_5, {}^4\!/_5] \Rightarrow v_2(A_2) = {}^3\!/_5$
- $A_3 = [{}^4\!/_5, 1] \Rightarrow v_3(A_3) = {}^3\!/_5$

- The allocation is proportional, envy-free, and equitable

# Relations Between Fairness Desiderata

- Envy-freeness implies proportionality
  - Summing $v_i(A_i) \geq v_i(A_j)$ over all $j$ gives proportionality
- For 2 agents, proportionality also implies envy-freeness
  - Hence, they are equivalent.
- Equitability is incomparable to proportionality and envy-freeness
  - E.g. if each agent has value 0 for her own allocation and 1 for the other agent's allocation, it is equitable but not proportional or envy-free.

# Existence

- Theorem [Alon, 1987]
  Suppose the value density function $f_i$ of each agent valuation $v_i$ is continuous. Then, we can cut the cake at $n^2 - n$ places and rearrange the $n^2 - n + 1$ intervals into $n$ pieces $A_1, \dots, A_n$ such that
  $$v_i(A_j) = {}^1/_n, \forall i, j \in N$$

- This is called a "perfect partition"
  - It is trivially envy-free (thus proportional) and equitable
- As we will later see, this cannot be found with finitely many queries in Robertson-Webb model

# Proportionality

# PROPORTIONALITY : $n = 2$ AGENTS

- CUT-AND-CHOOSE
  - Agent 1 cuts the cake at $x$ such that $v_1([0, x]) = v_1([x, 1]) = 1/2$
  - Agent 2 chooses the piece that she prefers.

- Elegant protocol
  - Proportional (equivalent to envy-freeness for 2 agents)
  - Needs only one cut and one eval query (optimal)

- More agents?

# Proportionality: Dubins-Spanier

- Dubins-Spanier
  - Referee starts a knife at $0$ and moves the knife to the right.
  - Repeat: When the piece to the left of the knife is worth $1/n$ to an agent, the agent shouts "stop", receives the piece, and exits.
  - When only one agent remains, she gets the remaining piece.

- Can be implemented easily in Robertson-Webb model
  - When $[x, 1]$ is left, ask each remaining agent $i$ to cut at $y_i$ so that $v_i([x, y_i]) = 1/n$, and give agent $i^* \in \arg\min_i y_i$ the piece $[x, y_{i^*}]$.
- Query complexity: $\Theta(n^2)$

# PROPORTIONALITY: EVEN-PAZ

- EVEN-PAZ
- Input:
  - Interval $[x, y]$, number of agents $n$ (assume a power of 2 for simplicity)
- Recursive procedure:
  - If $n = 1$, give $[x, y]$ to the single agent.
  - Otherwise:
    - Each agent $i$ marks $z_i$ such that $v_i([x, z_i]) = v_i([z_i, y])$
    - $z^*$ = $(n/2)^{\text{th}}$ mark from the left.
    - Recurse on $[x, z^*]$ with the left $n/2$ agents, and on $[z^*, y]$ with the right $n/2$ agents.

- Query complexity: $\Theta(n \log n)$

# Complexity of Proportionality

- Theorem [Edmonds and Pruhs, 2006]:
  - Any protocol returning a proportional allocation needs $\Omega(n \log n)$ queries in the Robertson-Webb model.

- Hence, EVEN-PAZ is provably (asymptotically) optimal!

# Envy-Freeness

# Envy-Freeness : Few Agents

- $n = 2$ agents : CUT-AND-CHOOSE (2 queries)
- $n = 3$ agents : SELFRIDGE-CONWAY (14 queries)

Gets complex pretty quickly!

Suppose we have three players **P1**, **P2** and **P3**. Where the procedure gives a criterion for a decision it means that criterion gives an optimum choice for the player.

1. **P1** divides the cake into three pieces he considers of equal size.
2. Let's call **A** the largest piece according to **P2**.
3. **P2** cuts off a bit of **A** to make it the same size as the second largest. Now **A** is divided into: the trimmed piece **A1** and the trimmings **A2**. Leave the trimmings **A2** to the side for now.
    - If **P2** thinks that the two largest parts are equal (such that no trimming is needed), then each player chooses a part in this order: **P3**, **P2** and finally **P1**.
4. **P3** chooses a piece among **A1** and the two other pieces.
5. **P2** chooses a piece with the limitation that if **P3** didn't choose **A1**, **P2** must choose it.
6. **P1** chooses the last piece leaving just the trimmings **A2** to be divided.

It remains to divide the trimmings **A2**. The trimmed piece **A1** has been chosen by either **P2** or **P3**; let's call the player who chose it **PA** and the other player **PB**.

1. **PB** cuts **A2** into three equal pieces.
2. **PA** chooses a piece of **A2** - we name it **A21**.
3. **P1** chooses a piece of **A2** - we name it **A22**.
4. **PB** chooses the last remaining piece of **A2** - we name it **A23**.

# Envy-Freeness : Few Agents

- [Brams and Taylor, 1995]
  - The first finite (but unbounded) protocol for any number of agents

- [Aziz and Mackenzie, 2016a]
  - The first bounded protocol for $4$ agents (at most 203 queries)

- [Amanatidis et al., 2018]
  - A simplified version of the above protocol for $4$ agents (at most 171 queries)

# Envy-Freeness

- Theorem [Aziz and Mackenzie, 2016b]
  - There exists a bounded protocol for computing an envy-free allocation with $n$ agents, which requires $O(n^{n^{n^{n^{n^{n}}}}})$ queries
  - After $O(n^{2n+3})$ queries, the protocol can output a partial allocation that is both proportional and envy-free

- What about lower bounds?

# Complexity of Envy-Freeness

- Theorem [Procaccia, 2009]
  Any protocol for finding an envy-free allocation requires $\Omega(n^2)$ queries.

**Open Problem**

Bridge the gap between $O\left(n^{n^{n^{n^{n^{n}}}}}\right)$ upper bound and $\Omega(n^2)$ lower bound for envy-free cake-cutting

- Theorem [Stromquist, 2008]
  There is no finite (even unbounded) protocol for finding a simple envy-free allocation for $n \geq 3$ agents.

# Equitability

# Upper Bound: $n = 2$ Agents

- Existence
  - Suppose we cut the cake at $x$ to form pieces $[0, x]$ and $[x, 1]$
  - Let $f(x) = v_1([0, x]) - v_2([x, 1])$
    - Note that $f(0) = -1$, $f(1) = 1$, and $f$ is continuous
  - By the intermediate value theorem: $\exists x^*$ such that $f(x^*) = 0$
  - Allocation $A_1 = [0, x^*]$ and $A_2 = [x^*, 1]$ is equitable

- Theorem [Cechlárová and Pillárová, 2012]
  - Using binary search for $x^*$, we can find an $\epsilon$-equitable allocation for 2 agents with $O(\ln(1/\epsilon))$ queries.

# Upper Bound: $n > 2$ Agents

- Theorem [Cechlárová and Pillárová, 2012]
  - This technique can be extended to $n$ agents to find an $\epsilon$-equitable allocation in $O(n \ln(^1/_\epsilon))$ queries.

- Theorem [Procaccia and Wang, 2017]
  - There exists a protocol for $n$ agents which finds an $\epsilon$-equitable allocation in $O(^1/_\epsilon \ \ln(^1/_\epsilon))$ queries.
  - Intuition:
    - If $n \leq {}^1/_\epsilon$, use above protocol for finding an equitable $\epsilon$-equitable allocation.
    - If $n > {}^1/_\epsilon$, use a variant of the Evan-Paz algorithm to find an *anti-proportional* allocation where $n' = \lceil ^1/_\epsilon \rceil$ agents get value *at most* $1/n'$, and the rest receive nothing.
      - While this is a "bad" allocation, it is $\epsilon$-equitable.

# Lower Bound

- Theorem [Procaccia and Wang, 2017]
  Any protocol for finding an $\epsilon$-equitable allocation must require $\Omega\left(\frac{\ln(1/\epsilon)}{\ln\ln(1/\epsilon)}\right)$ queries.

- Theorem [Procaccia and Wang, 2017]
  There is no finite (even if unbounded) protocol for finding an equitable allocation.
  - Non-existence of bounded protocols follows from the previous result.
  - But their proof works for non-existence of unbounded protocols as well.

# Price of Fairness

# Price of Fairness

- Measures the worst-case loss in social welfare due to requirement of a fairness property $X$
- Social welfare of allocation $A$ is the sum of values of the agents
  - Denoted $sw(A) = \sum_{i \in N} v_i(A_i)$
- Let $\mathcal{F}$ denote the set of feasible allocations and $\mathcal{F}_X$ denote the set of feasible allocations satisfying property $X$

$$PoF_X = \sup_{v_1,\ldots,v_n} \frac{\max_{A \in \mathcal{F}} sw(A)}{\max_{A \in \mathcal{F}_X} sw(A)}$$

# Price of Fairness

- Theorem [Caragiannis et al., 2009]
  For cake-cutting, the price of proportionality is $\Theta(\sqrt{n})$, and the price of equitability is $\Theta(n)$.

- Theorem [Bertsimas et al., 2011]
  For cake-cutting, the price of envy-freeness is also $\Theta(\sqrt{n})$. This is achieved by an allocation maximizing the Nash welfare $\Pi_i \, v_i(A_i)$.

  - Fun fact: The price of EF in cake-cutting was mentioned as an open question in a previous version of this tutorial, and was also believed to be open by many groups of researchers until recently.

# Efficiency

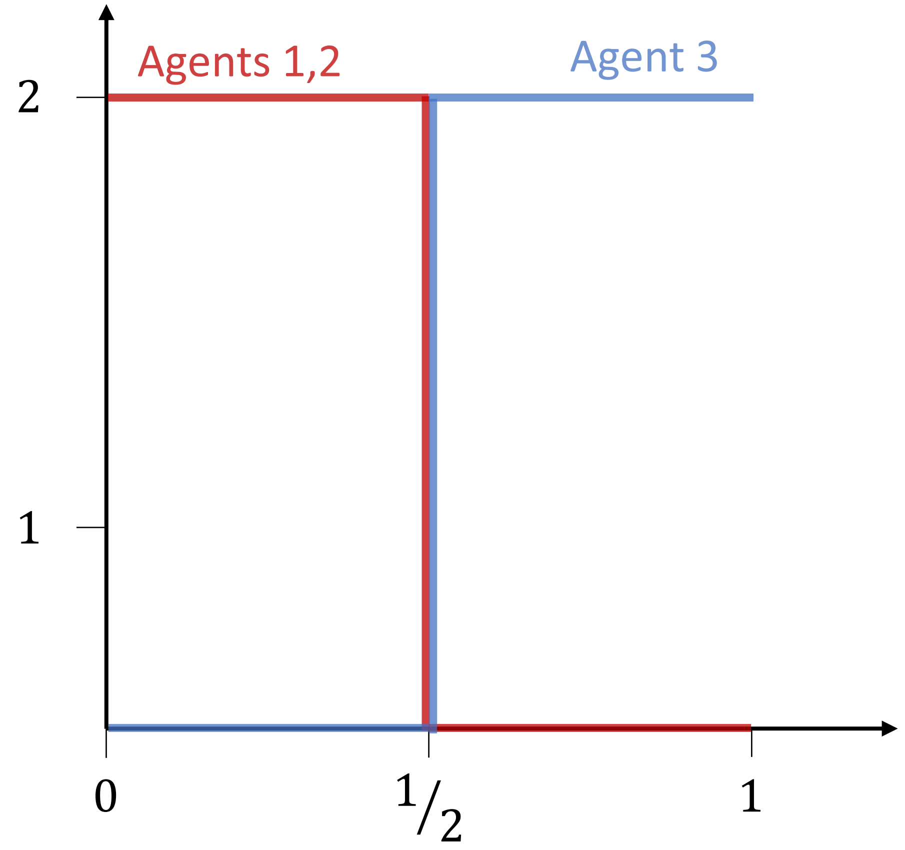
# Efficiency

- Weak Pareto optimality (WPO)
  - Allocation $A$ is weakly Pareto optimal if there is no allocation $B$ such that $v_i(B_i) > v_i(A_i)$ for all $i \in N$.
  - "Can't make everyone happier"
- Pareto optimality (PO)
  - Allocation $A$ is Pareto optimal if there is no allocation $B$ such that $v_i(B_i) \geq v_i(A_i)$ for all agents $i \in N$, and at least one inequality is strict.
  - "Can't make someone happier without making someone else less happy"
  - Easy to achieve in isolation (e.g. "serial dictatorship")

# PO+EF+EQ: (Non-)Existence

- Theorem [Barbanel and Brams, 2011]
  With two agents, there always exists an allocation that is envy-free (thus proportional), equitable, and Pareto optimal.
  - Their algorithm has similarities to the more popular "adjusted winner" algorithm, which we will see later in the tutorial.

- With $n \geq 3$ agents, PO+EQ is impossible



# What about PO+EF?

- Competitive Equilibrium from Equal Incomes (CEEI)
  - At equilibrium: there is an additive price function $P$ on the cake, and each agent gets to buy their best piece from a budget of one unit of fake currency
  - WCE: $\forall i \in N, Z \subseteq [0,1]: P(Z) \leq P(A_i) \Rightarrow v_i(Z) \leq v_i(A_i)$
  - EI: $\forall i \in N: P(A_i) = 1$

- Theorem [Weller, 1985]
  For cake-cutting, a CEEI always exists. Every CEEI is both envy-free and weakly Pareto optimal.

# s-CEEI

- Strong Competitive Equilibrium from Equal Incomes (s-CEEI)
  - A positive slice $Z$ is a subset of the cake valued positively by at least one agent
  - Allocation $A$ is called s-CEEI allocation if there exists an additive price function $P$ satisfying
  - $P(Z) > 0$ iff $Z$ is a positive slice
  - SCE: $\forall i \in N$, and positive slices $Z \subseteq [0,1]$ and $Z_i \subseteq A_i$: $\frac{v_i(Z_i)}{P(Z_i)} \geq \frac{v_i(Z)}{P(Z)}$
  - EI: $\forall i \in N$: $P(A_i) = 1$

Maximum bang-per-buck

- Theorem [Segal-Halevi and Sziklai, 2018]
  For cake-cutting, an s-CEEI allocation always exists. Every s-CEEI allocation is envy-free and Pareto optimal.

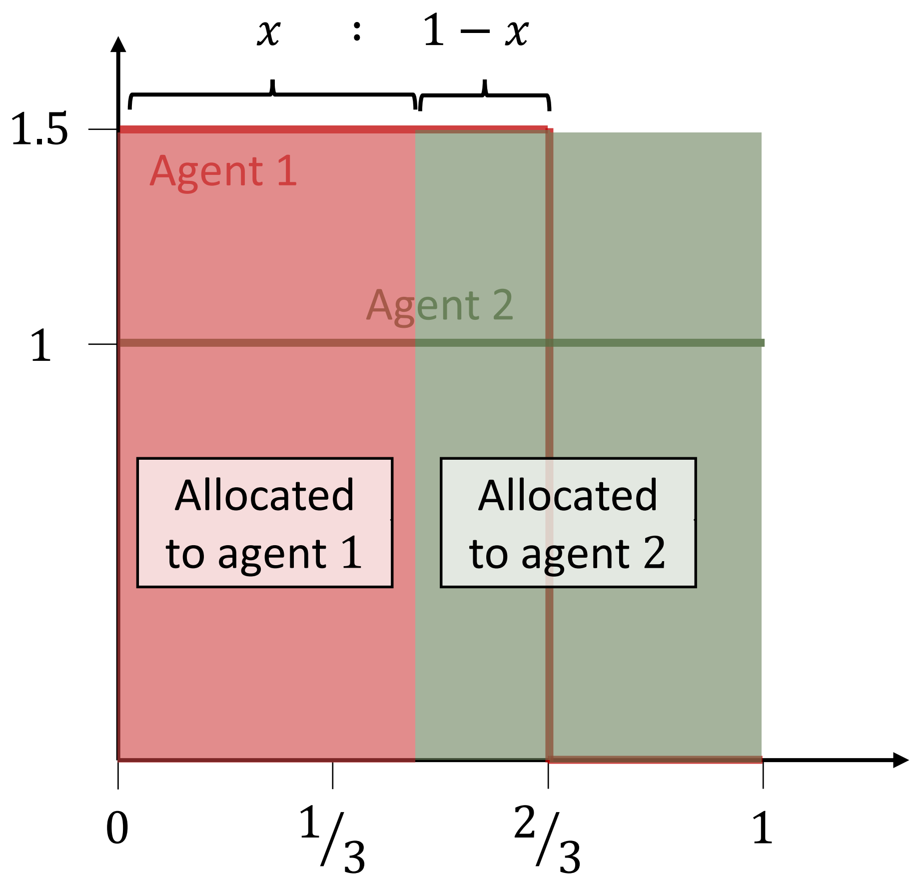
# s-CEEI and Nash-Optimality

- An allocation $A^*$ is called Nash-optimal if

$$A^* \in \arg\max_A \Pi_{i \in N} v_i(A_i)$$

- Theorem [Segal-Halevi and Sziklai, 2018]
  For cake-cutting, the set of s-CEEI allocations is exactly the same as the set of Nash-optimal allocations.

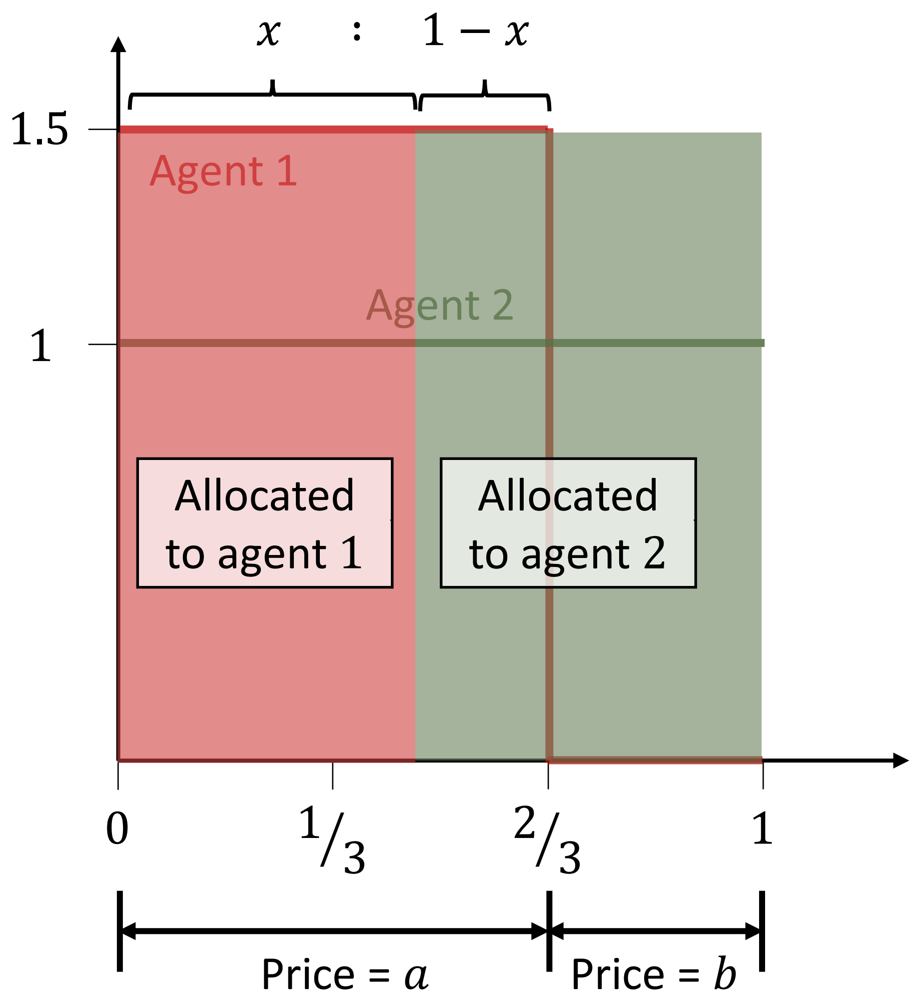
# Nash-Optimality Example




- Due to PO, suppose:
  - Agent 1 gets $x$ fraction of $[0, {}^2/_3]$
  - Agent 2 gets $1 - x$ fraction of $[0, {}^2/_3]$ and all of $[{}^2/_3, 1]$
  - $v_1(A_1) = x$
  - $v_2(A_2) = (1 - \mathrm{x}) \cdot {}^2/_3 + {}^1/_3 = {}^{(3-2\mathrm{x})}/_3$

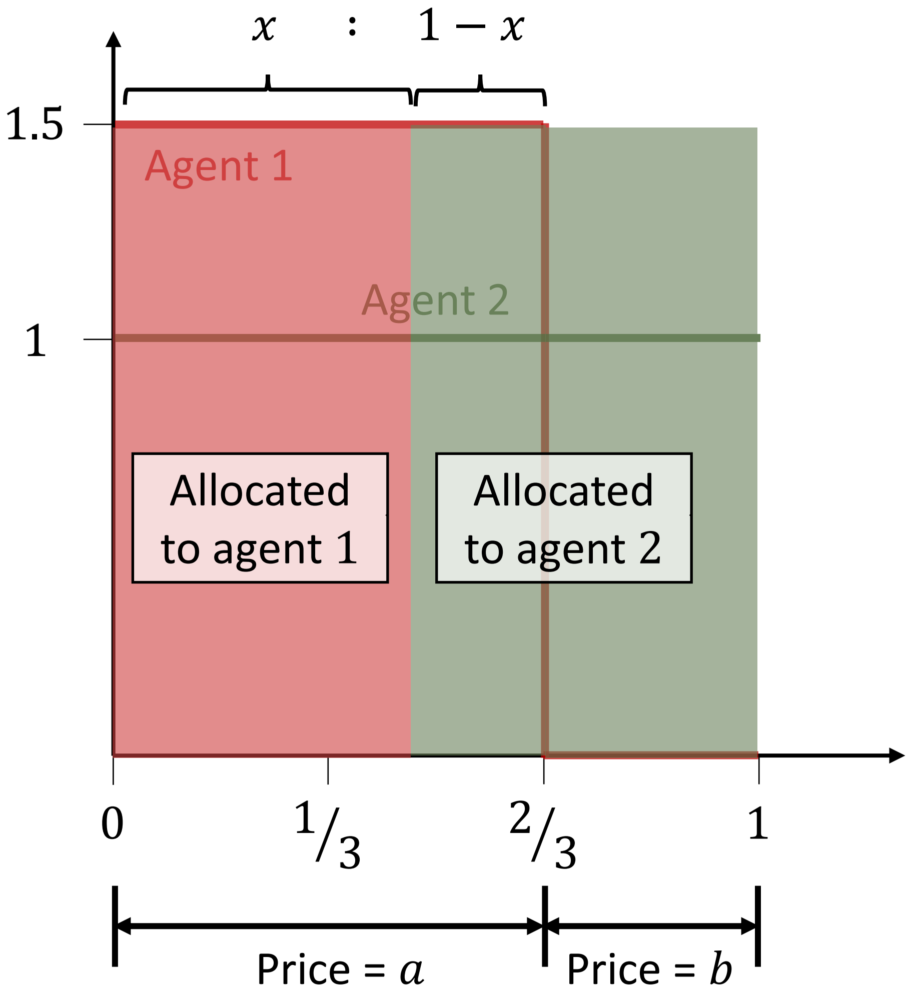
- Maximize $x \cdot {}^{(3-2\mathrm{x})}/_3 \Rightarrow x = {}^3/_4$
  - Nash-optimal allocation:
    - $A_1 = [0, {}^1/_2],\ v_1(A_1) = {}^3/_4$
    - $A_2 = [{}^1/_2, 1],\ v_2(A_2) = {}^1/_2$

# Nash-Optimality = s-CEEI




- Still must be PO, so like before
  - Agent 1 buys $x$ fraction of $[0, 2/3]$
  - Agent 2 buys $1 - x$ fraction of $[0, 2/3]$ and all of $[2/3, 1]$
- Prices: $P([0, 2/3]) = a, P([2/3, 1]) = b$
  - Spending: $a \cdot x = 1$, $a \cdot (1 - x) + b = 1$
    - Hence, $a + b = 2$
- Two cases: $x < 1$ or $x = 1$

# Nash-Optimality = s-CEEI

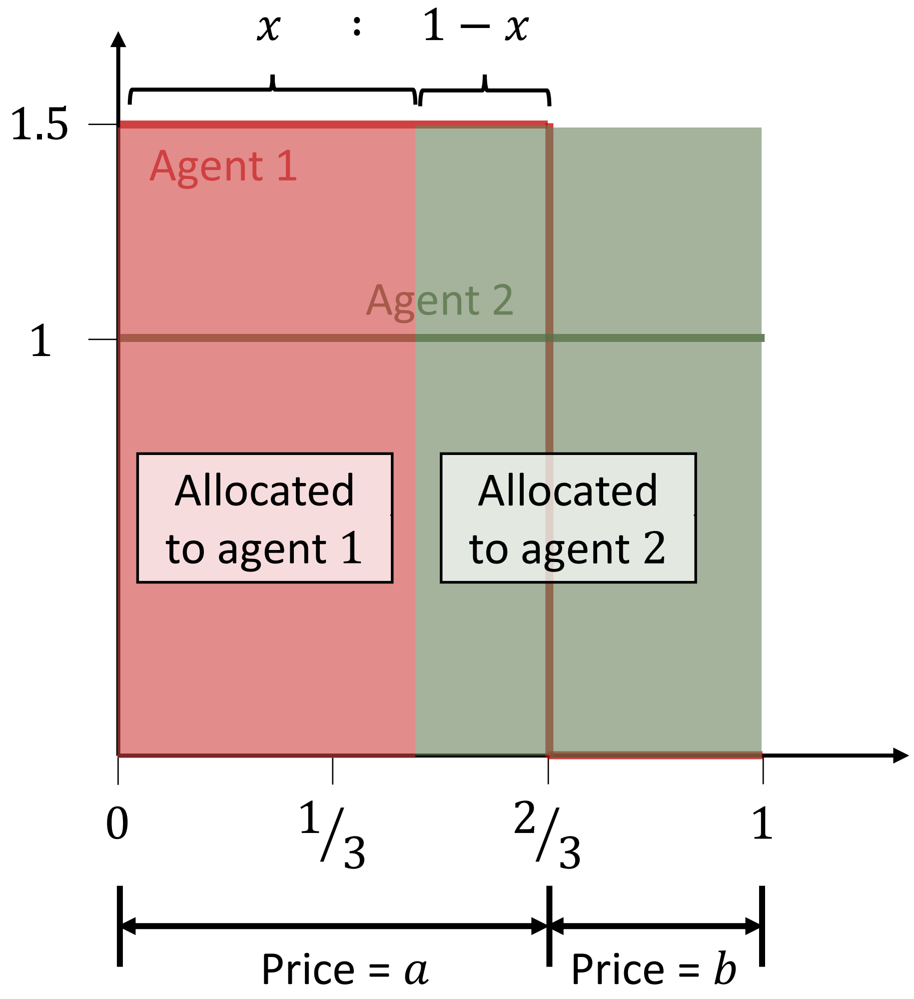



- $x < 1$
  - Agent 2 buys parts of both pieces
  - MBB:

$$\frac{1/3}{b} = \frac{2/3}{a} \;\Rightarrow\; a = 2b \;\Rightarrow\; (a,b) = \left(4/3, 2/3\right)$$

  - Substituting in $a \cdot x = 1$, we get $x = 3/4$
    - Same as Nash-optimal solution

# Nash-Optimality = s-CEEI




- $x = 1$
  - Since $a \cdot x = 1, a \cdot (1-x) + b = 1$, we get that $a = b = 1$
  - Agent 2 buys the second piece, so by MBB:

$$\frac{^1/_3}{b} \geq \frac{^2/_3}{a} \Rightarrow a \geq 2b$$

  - Contradiction!
  - So there is no s-CEEI with $x = 1$

# Strategyproofness

# Strategyproofness (SP)

- Direct-revelation mechanisms
  - A direct-revelation mechanism $h$ takes as input all the valuation functions $v_1, \dots, v_n$, and returns an allocation $A$
  - Notation: $h(v_1, \dots, v_n) = A$, $h_i(v_1, \dots, v_n) = A_i$

- Strategyproofness (deterministic mechanisms)
  - A direct-revelation mechanism $h$ is called strategyproof if

$$\forall v_1, \dots, v_n, \forall i, \forall v_i' : v_i\big(h_i(v_1, \dots, v_n)\big) \geq v_i(h_i(v_1, \dots, v_i', \dots, v_n))$$

  - That is, no agent $i$ can achieve a higher value by misreporting her valuation, regardless of what the other agents report

# Strategyproofness (SP)

- Strategyproofness (randomized mechanisms)
  - Technically, referred to as "truthfulness-in-expectation"
    - When referring to SP for randomized mechanisms, we will refer to this concept
  - A randomized direct-revelation mechanism $h$ is called strategyproof if

    $$\forall v_1, \dots, v_n, \forall i, \forall v_i' : E\left[v_i\left(h_i(v_1, \dots, v_n)\right)\right] \geq E\left[v_i\left(h_i(v_1, \dots, v_i', \dots, v_n)\right)\right]$$

  - That is, no agent $i$ can achieve a higher *expected* value by misreporting her valuation, regardless of what the other agents report
    - Expectation is over the randomness of the mechanism

# Deterministic SP Mechanisms

- Theorem [Menon and Larson '17, Bei et al. '17]
  No non-wasteful deterministic SP mechanism is (even approximately) proportional.
  - Since EF is at least as strict as Prop, SP+EF is also impossible subject to non-wastefulness.
  - Non-wastefulness can be replaced by a requirement of "connected pieces", and the impossibility result still holds.

**Open Problem**

Does the SP+Prop impossibility hold
even without the non-wastefulness assumption?

# Deterministic SP Mechanisms

- SP+PO is easy to achieve
  - E.g. serial dictatorship

- SP+PO+EQ is impossible
  - We saw that even EQ+PO allocations may not exist

**Open Problem**

Does there exist a direct revelation, deterministic SP+EQ mechanism?

# Randomized SP Mechanisms

- We want the mechanism *always* return an allocation satisfying a subset of {EQ,EF,PO}, and be SP in expected utilities

- Recall: PO+EQ allocations may not exist
  - Hence, we can only hope for SP+PO+EF or SP+EF+EQ
  - The first is an open problem, but the second combination is achievable!

**Open Problem**

Does there exist a randomized SP mechanism which always returns a PO+EF allocation?

# Randomized SP Mechanisms

- Theorem [Mossel and Tamuz, 2010; Chen et al. 2013]
  There is a randomized SP mechanism that *always* returns an EF+EQ allocation.
  - Recall: In a perfect partition $B$, $v_i(B_k) = {}^1\!/_n$ for all $i, k \in N$
  - Algorithm: Compute a perfect partition and return allocation $A$ which randomly assigns the $n$ pieces to the $n$ agents
  - SP: Regardless of what the agents report, agent $i$ receives each piece of the cake with probability $1/n$, and thus has expected value exactly $1/n$
  - EF: Assuming agents report truthfully (due to SP), agent $i$ always receives a cake she values at $1/n$, and according to her, so do others.

# Existential Summary

✗ = Impossibility
✓ = Possibility

**SP+PO+EF+EQ**
✗ Rand

**SP+PO+EF**
✗ Det
? Rand

**SP+PO+EQ**
✗ Rand

**SP+EF+EQ**
✗ Det
✓ Rand

**PO+EF+EQ**
✗ Rand

**SP+PO**
✓ Det

**SP+EF**
✗ Det
✓ Rand

**SP+EQ**
? Det
✓ Rand

**PO+EF**
✓ Det

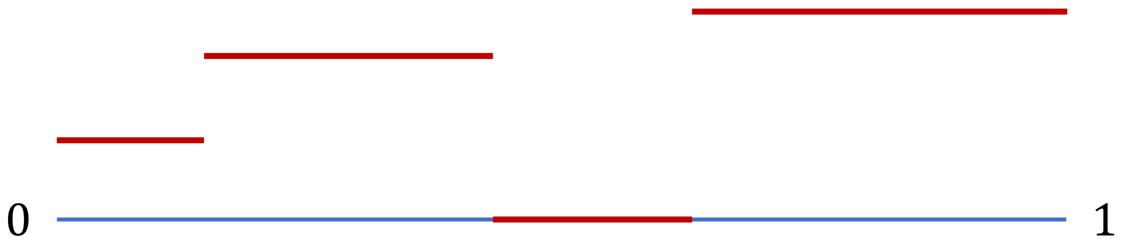
**PO+EQ**
✗ Rand

**EF+EQ**
✓ Det

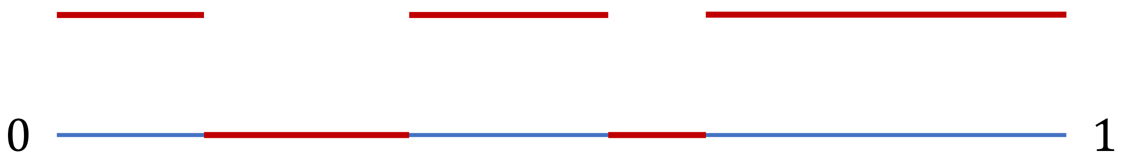
# Special Cases

# Piecewise Constant/Uniform Valuations

Piecewise constant density function




Piecewise uniform density function




Special case of piecewise constant

# Possibilities

- Theorem [Chen et al., 2013]
  For piecewise uniform valuations, there exists a deterministic SP mechanism which returns an EF+PO allocation.
  - Recall that for general valuations, even deterministic SP+EF is impossible.

- Theorem [Aziz and Ye, 2014]
  For piecewise constant valuations, an s-CEEI (i.e. Nash-optimal) allocation can be computed in polynomial time.
  - Recall that this is EF (thus Prop) and PO.
  - But this is not SP.

# EF in Robertson-Webb

- Theorem [Kurokawa et al., 2013]
  If an algorithm computes an envy-free allocation for $n$ agents with piecewise uniform valuations with at most $g(n)$ queries, then it can also compute an envy-free allocation for $n$ agents with general valuations with at most $g(n)$ queries.

  - Let the same algorithm interact with general valuations $v_1, \dots, v_n$ via CUT and EVAL queries and return an allocation $A$
  - The proof constructs piecewise uniform valuations $u_1, \dots, u_n$ which would have resulted in the same responses and $u_i(A_j) = v_i(A_j)$ for all $i, j \in N$

# PO in Robertson-Webb

- Non-wastefulness
  - An allocation $A$ is called non-wasteful if no piece of the cake that is valued positively by at least one agent is assigned to an agent who has zero value for it
  - PO implies non-wastefulness

- Theorem [Ianovski, 2012; Kurokawa et al., 2013]
  No finite protocol in the Robertson-Webb model can always produce a non-wasteful allocation, even for piecewise uniform valuations.

- This is the reason we did not provide query complexity results when discussing PO

# Burnt Cake Division

# Model

- Same as regular cake, except agents now have non-positive valuation for every piece of the cake
  - $f_i(x) \leq 0, \forall x \in [0,1]$
  - Hence, $v_i(X) \leq 0, \forall X \in \mathcal{A}$

- Equitability and perfect partitions carry over from the goods case
  - Simply use $-f_i$ and $-v_i$

# Dividing a Burnt Cake

- Theorem [Peterson and Su, 2009]
  For burnt cake division, there exists a finite (but unbounded) protocol for finding an envy-free allocation with $n$ agents.
  - Builds upon the Brams-Taylor protocol for dividing a good cake
  - But certain operations require non-trivial transformations to the world of chores

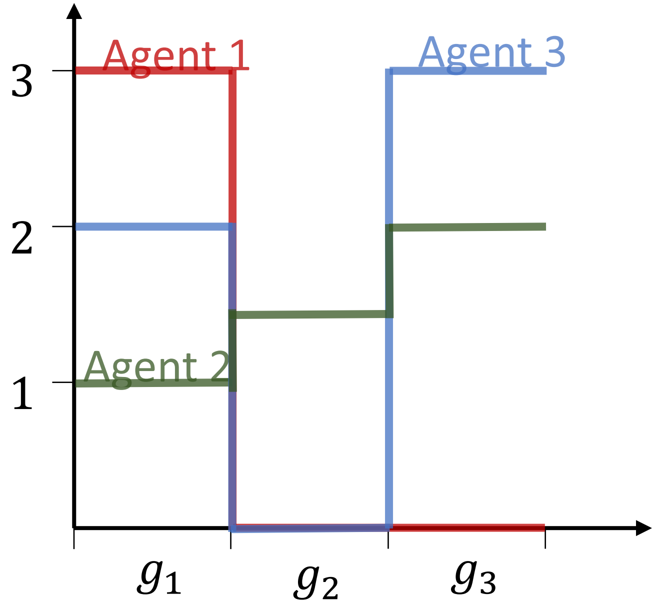
**Open Problem**

Is there a bounded envy-free protocol for burnt cake division?

# (Homogeneous) Divisible Goods

# Model

- Agents: $N = \{1, 2, \dots, n\}$
- Resource: Set of divisible goods $M = \{g_1, g_2, \dots, g_m\}$
- Allocation $A = (A_1, \dots, A_n)$
  - $A_i = \left(A_{i,j}\right)_{j \in [m]}$
  - $\forall i, j$: $A_{i,j} \in [0,1]$
  - $\forall j$: $\sum_i A_{i,j} \leq 1$




- Assume additive valuations $v_i(A_i) = \sum_j A_{i,j} v_i(g_j)$
- Special case of cake cutting (up to normalization)

# $n = 2$: Adjusted Winner Procedure

[Brams and Taylor 1996]

- Input: Normalized valuation functions
- Order the goods by ratio $v_1(g)/v_2(g)$.

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| $a_1$ | 20 | 30 | 15 ~~30~~ | 10 | 5 | 5 |
| $a_2$ | 10 | 15 | ~~20~~ 10 | 15 | 10 | 30 |

$v_1(g)/v_2(g)$ high ⟷ $v_1(g)/v_2(g)$ low

- Divide the goods so that agent 1 receives goods $g_1, \ldots, g_{j-1}$, agent 2 receives goods $g_{j+1}, \ldots, g_m$ for some $j$, and $v_1(A_1) = v_2(A_2)$
  - $g_j$ is divided between the agents, if necessary

# $n = 2$: Adjusted Winner Procedure

[Brams and Taylor 1996]

- Theorem [Brams and Taylor 1996]:
  - The adjusted winner procedure is envy-free (and therefore proportional), equitable and Pareto optimal

- Breaks down for $n > 2$
  - As in cake cutting, EF + EQ + PO is impossible, what about two of the three?
  - EF+EQ: Divide each good equally among agents (“perfect partition”)
  - EQ + PO: Impossible
  - EF + PO: Can achieve with CEEI

| | $g_1$ | $g_2$ |
|---|---|---|
| $\boldsymbol{a_1}$ | 1 | 0 |
| $\boldsymbol{a_2}$ | 1 | 0 |
| $\boldsymbol{a_3}$ | 0 | 1 |

# CEEI

- With a fixed set of items, the definition of s-CEEI (that we will now call just CEEI) becomes simpler.

- Equilibrium price $p_j > 0$ for each good $g_j$
  - Assume for simplicity that $\forall j\ \exists i$ with $v_i(g_j) > 0$
- CE: If $A_{i,j} > 0$ then $\frac{v_i(g_j)}{p_j} \geq \frac{v_i(g_k)}{p_k}$ for all $k$
- EI: $\sum_j p_j A_{i,j} = 1$ for all $i$

# Eisenberg-Gale convex program

- Can compute a CEEI allocation as the solution to the Eisenberg-Gale [1959] convex program:

$$\max \sum_{i \in N} \log u_i \;\; s.t.$$

$$\forall i: u_i \leq \sum_{g_j \in M} A_{i,j} v_i(g_j)$$

$$\forall j: \sum_{i \in N} A_{i,j} \leq 1$$

$$\forall i, j: A_{i,j} \geq 0$$

- Theorem [Orlin 2010, Végh 2012]:
  - The Eisenberg-Gale convex program can be solved in strongly polynomial time.

# Strategyproofness

- CEEI solution is fair and efficient but not strategyproof.
  - It is strategyproof in the large (SP-L) [Azevedo and Budish 2018] though
- Theorem [Han et al. 2011]:
  - No strategyproof mechanism that always outputs a complete allocation can achieve better than a ${}^{1}/_{m}$ approximation to the optimal social welfare for large enough $n$.
    - Social welfare = $\sum_{i \in N} v_i(A_i)$
- Theorem [Cole et al. 2013]:
  - There is a strategyproof partial allocation mechanism that provides every agent with a $1/e$ fraction of their CEEI utility.
  - Allocation is envy-free but not proportional

# SP + Prop + EF

- SP + Prop + EF is trivial! Just allocate everyone an equal fraction of each good.
  - What if we also want PO?
- Theorem [Schummer 1996]:
  - It is impossible to achieve SP + Prop + PO.
  - SP + PO: Serial dictatorship.
- SP + Prop + EF can also be achieved non-trivially [Freeman et al. 2019]
  - Additionally achieves strict SP: agents always achieve strictly higher utility by reporting their beliefs truthfully than by lying.
  - Exploits a correspondence between fair division and wagering mechanisms [Lambert et al. 2008] to utilize proper scoring rules (e.g. Brier score)

# Allocating Divisible Goods + Bads

# Model

- Agents: $N = \{1, 2, \dots, n\}$
- Resources: Set of divisible "items" $M = \{o_1, o_2, \dots, o_m\}$
- Allocation $A = (A_1, \dots, A_n)$
  - $A_i = \left(A_{i,j}\right)_{j \in [m]}$
  - $\forall i, j$: $A_{i,j} \in [0,1]$
  - $\forall j$: $\sum_i A_{i,j} \leq 1$
- Assume additive valuations: $v_i(A_i) = \sum_j A_{i,j} v_i(o_j)$
  - However, $v_i(o_j)$ can be positive, zero, or negative
- We'll refer to s-CEEI simply as CEEI in this case

# Achieving EF+PO

- Theorem [Bogomolnaia et al. 2017]
  - ➢ There always exists a CEEI allocation, which is envy-free and Pareto optimal.
  - ➢ The CEEI solution is "welfarist", i.e., the set of feasible utility profiles is enough to identify the set of CEEI utility profiles.
  - ➢ The CEEI utility profile is given by the following:
    1. If it is possible to give a positive utility to each agent (who can receive a positive utility), then maximizing the Nash welfare gives the unique CEEI utility profile.
    2. Else, if the all-zero utility profile is feasible and Pareto optimal, then it is the unique CEEI utility profile.
    3. Else, there can be exponentially many CEEI utility profiles, which give non-positive utility to each agent.
  - ➢ Their actual result is stronger and in a more general model

# Not Covered

- Nash equilibria of cake-cutting
- Optimal cake-cutting
  - Algorithms for maximizing social welfare subject to fairness constraints
- Number of cuts and moving knives protocols
  - Possibility and impossibility results for $n - 1$ cuts
- Multidimensional cakes
- Randomized or strategyproof Robertson-Webb protocols
- Non-additive valuations
- …

# Indivisible Goods

# Model

- Agents: $N = \{1, 2, \dots, n\}$

- Resource: Set of indivisible goods $M = \{g_1, g_2, \dots, g_m\}$

- Allocation $A = (A_1, \dots, A_n) \in \Pi_n(M')$ is a partition of $M'$ for some $M' \subseteq M$.

- Each agent $i$ has a valuation $v_i : 2^M \to \mathbb{R}_+$
  - $v_i : 2^M \to \mathbb{R}_-$ in the case of bads, $v_i : 2^M \to \mathbb{R}$ for both goods and bads

# Valuation Functions

- Additive: $\forall X, Y$ with $X \cap Y = \emptyset$: $v_i(X \cup Y) = v_i(X) + v_i(Y)$
  - Equivalently: $v_i(X) = \sum_{g \in X} v_i(g)$
  - Value for a good independent of other goods received

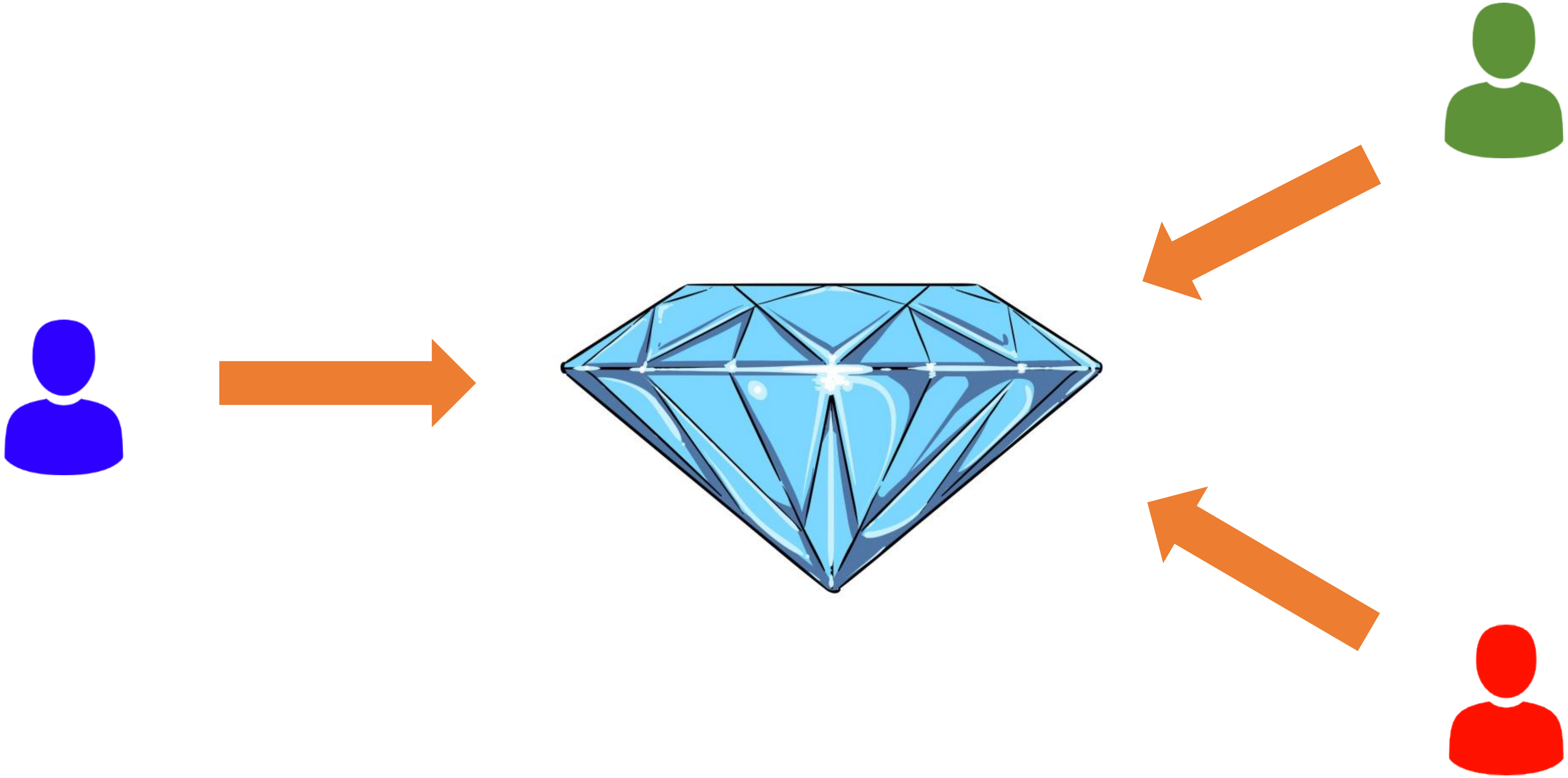
Most results for additive valuations unless stated otherwise

- Submodular: $\forall X, Y$ : $v_i(X \cup Y) + v_i(X \cap Y) \leq v_i(X) + v_i(Y)$
  - Equivalently: $\forall X, Y$ with $X \subseteq Y$: $v_i(X \cup \{g\}) - v_i(X) \geq v_i(Y \cup \{g\}) - v_i(Y)$

- Subadditive: $\forall X, Y$ with $X \cap Y = \emptyset$: $v_i(X \cup Y) \leq v_i(X) + v_i(Y)$

- Submodular and subadditive definitions capture the idea of diminishing returns.

# Need new guarantees!



# Envy-Freeness up to One Good

# Envy-Freeness up to One Good (EF1)

[Lipton et al 2004, Budish 2011]

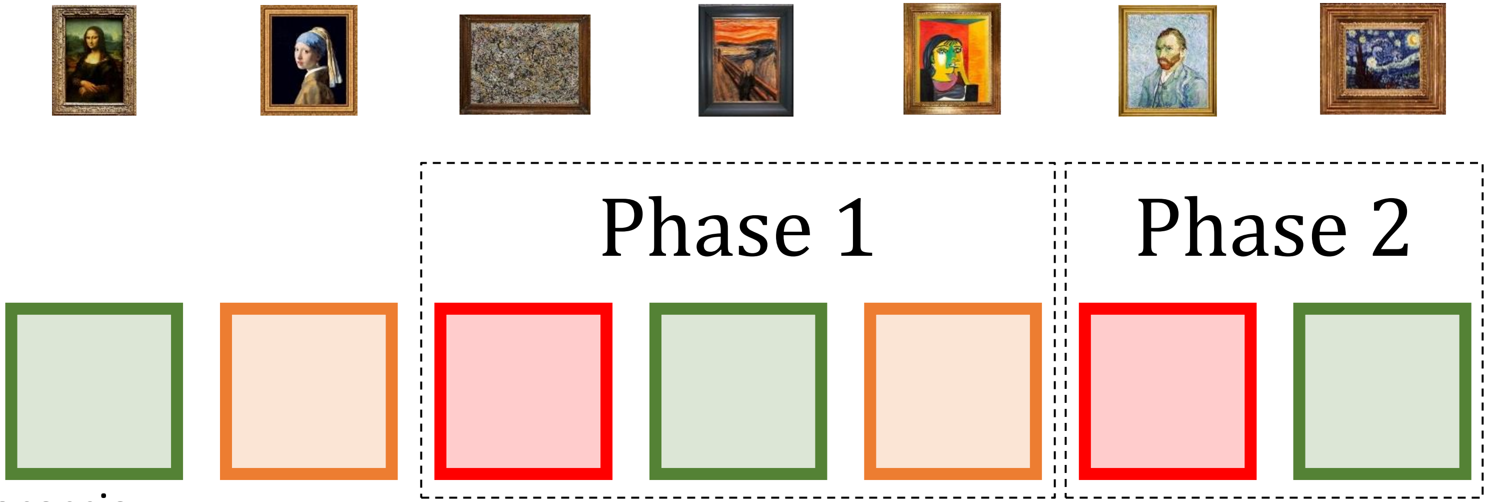
- An allocation is envy-free up to one good (EF1) if, for all agents $i, j$, there exists a good $g \in A_j$ for which

$$v_i(A_i) \geq v_i(A_j \setminus \{g\})$$

- "Agent $i$ may envy agent $j$, but the envy can be eliminated by removing a single good from $j$'s bundle."
  - Note: We don't consider $A_j = \emptyset$ a violation of EF1.

# Round Robin Algorithm

- Fix an ordering of the agents $\sigma$.
- In round $k \bmod n$, agent $\sigma_k$ selects their most preferred remaining good.
- Theorem: Round robin satisfies EF1.




Animation Credit: Ariel Procaccia

# Algorithm for Achieving EF1

- Envy graph: Edge from $i$ to $j$ if $i$ envies $j$

- Greedy algorithm [Lipton et al. 2004]
  - ➢ One at a time, allocate a good to an agent that no one envies
  - ➢ While there is an envy cycle, rotate the bundles along the cycle.
    - Can prove this loop terminates in a polynomial number of steps

- Removing the most recently added good from an agent's bundle removes envy towards them.
- Neither this algorithm nor round robin is Pareto optimal.

# EF1 with Goods and Bads [Aziz et al. 2019]

- An allocation is envy-free up to one item (EF1) if, for all agents $i, j$, there exists an item $o \in A_i \cup A_j$ for which

$$v_i(A_i \setminus \{o\}) \geq v_i(A_j \setminus \{o\})$$

- Round robin fails EF1

| | $o_1$ | $o_2$ | $o_3$ | $o_4$ |
|---|---|---|---|---|
| $a_1$ | 2 | 1 | -4 | -4 |
| $a_2$ | 2 | -3 | -4 | -4 |

# Double Round Robin

- Let $O^- = \{o \in O : \forall i \in N, v_i(o) \leq 0\}$ denote all unanimous bads and $O^+ = \{o \in O : \exists i \in N, v_i(o) > 0\}$ denote all objects that are a good for some agent.
  - Suppose that $|O^-| = an$ for some $a \in \mathbb{N}$. If not, add dummy bads with $v_i(o) = 0$ for all $i \in N$.

- Double round robin:
  - Phase 1: $O^-$ is allocated by round robin in order $(1, 2, \ldots, n-1, n)$
  - Phase 2: $O^+$ is allocated by round robin in order $(n, n-1, \ldots, 2, 1)$
  - Agents can choose to skip their turn in phase 2

# Double Round Robin

- Theorem [Aziz et al. 2019]:
  - The double round robin algorithm outputs an allocation that is EF1 for combinations of goods and bads in polynomial time.
  - Proof idea: Let $i < j$. Agent $i$ can envy $j$ up to one item in phase 1 (but not vice versa), and agent $j$ can envy $i$ up to one item in phase 2 (but not vice versa)

| | $O^+$ | | $O^-$ | |
|---|---|---|---|---|
| | $\boldsymbol{o_1}$ | $\boldsymbol{o_2}$ | $\boldsymbol{o_3}$ | $\boldsymbol{o_4}$ |
| $\boldsymbol{a_1}$ | 2 | 1 | -4 | -4 |
| $\boldsymbol{a_2}$ | 2 | -3 | -4 | -4 |

# Maximum Nash Welfare

- Maximum Nash Welfare (MNW): Select the allocation that maximizes the geometric mean of agent utilities (more on this later).

$$A = \arg\max \left( \prod_i v_i(A_i) \right)^{1/n}$$

  - ➢ This is just Nash-optimality from earlier

- What if $\prod_i v_i(A_i) = 0$ for all allocations?
  - ➢ Find an allocation that maximizes $|\{v_i(A_i) > 0\}|$, and subject to that maximizes

$$\left( \prod_{i:v_i(A_i)>0} v_i(A_i) \right)^{1/n}$$

# EF1 + PO

- Theorem [Caragiannis et al. 2016]:
  - The MNW allocation satisfies EF1 and PO.
  - PO: A Pareto-improving allocation would have higher geometric mean of utilities for agents with non-zero utility or more agents with non-zero utility.
  - EF1: Let $g_i^* = \arg\max_{g \in A_i} v_i(g)$. Not-too-hard proof shows $v_j(A_j) \geq v_j(A_i \setminus g_i^*)$ for all $j$.

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| $a_1$ | 2 | 1 | 3 | 0 | 1 | 2 |
| $a_2$ | 10 | 1 | 1 | 1 | 2 | 5 |
| $a_3$ | 3 | 1 | 3 | 0 | 5 | 2 |

# Computing EF1 + PO

- The MNW allocation is strongly NP-hard to compute (reduction from X3C).
  - Actually, it's APX-hard [Lee 2017].

- Special case: Binary valuations
  - MNW allocation can be computed in polynomial time [Darmann and Schauer 2015, Barman et al. 2018].
  - However, round robin already guarantees EF1 + PO in this setting.

# Computing EF1 + PO

- Theorem [Barman et al. 2018]:
  - There exists a pseudo-polynomial time algorithm for computing an allocation satisfying EF1 + PO
  - Algorithm uses local search (sequence of item swaps and price rises) to compute an integral competitive equilibrium that is price envy-free up to one good.
  - Price envy-free up to one good: $\forall i, k, \exists j: p(A_i) \geq p(A_k \setminus \{g_j\})$
  - Need different entitlements because CEEI might not exist with indivisibilities
    - Two agents, one item…

# Computing EF1 + PO

Open Problem:
Complexity of computing an EF1 + PO allocation

Open Problem:
Does there always exist an EF1 + PO allocation for submodular valuation functions?

# EF1 + PO for Bads

- Theorem [Aziz et al. 2019]:
  - When items can be either goods or bads and $n = 2$, an EF1 + PO allocation always exists and can be found in polynomial time

Open Problem:

Does an EF1 + PO allocation always exists for bads?

# Proportionality up to One Good

# Proportionality up to One Good (Prop1)

[Conitzer et al. 2017]

- An allocation is proportional up to one good (Prop1) if, for every agent $i$, there exists a good $g$ for which

$$v_i(A_i \cup \{g\}) \geq \frac{v_i(M)}{n}$$

| | $g_1$ | $g_2$ | $g_3$ |
|---|---|---|---|
| $a_1$ | 1 | 3 | 3 |
| $a_2$ | 1 | 3 | 3 |

$$v_1(A_1 \cup \{g_2\}) = 4 \geq \frac{7}{2} = \frac{v_i(M)}{n}$$

# Prop1 + PO

- Any algorithm that satisfies EF1 + PO is also Prop1 + PO.
  - MNW
  - Barman et al. [2018] algorithm
- Theorem [Barman and Krishnamurthy 2019]:
  - An allocation satisfying Prop1 + PO can be computed in strongly polynomial time.
- Allocation is a careful rounding of the fractional CEEI allocation.
  - In contrast, there exist instances in which no rounding of the fractional CEEI allocation will give EF1 [Caragiannis et al., 2016].

# Envy-Freeness up to the Least Valued Good

# Envy-Freeness up to the Least Valued Good

[Caragiannis et al. 2016]

| | $g_1$ | $g_2$ | $g_3$ |
|---|---|---|---|
| $a_1$ | 10 | 5 | 5 |
| $a_2$ | 10 | $\epsilon$ | $\epsilon$ |

- An allocation is envy-free up to the least valued good (EFX) if, for all agents $i, j$, and every $g \in A_j$ with $v_i(g) > 0$,

$$v_i(A_i) \geq v_i\left(A_j \setminus \{g\}\right).$$

# Leximin Allocation

- Leximin allocation:
  - First, maximize the minimum utility any agent receives. Subject to this, maximize the second-minimum utility. Then the third-minimum utility, etc.

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| $a_1$ | 2 | 1 | 3 | 0 | 1 | 2 |
| $a_2$ | 10 | 1 | 1 | 1 | 2 | 5 |
| $a_3$ | 3 | 1 | 3 | 0 | 5 | 2 |

# Satisfying EFX

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ |
|---|---|---|---|---|
| $a_1$ | 4 | 1 | 2 | 2 |
| $a_2$ | 4 | 1 | 2 | 2 |
| $a_3$ | 4 | 1 | 2 | 2 |

- Theorem [Plaut and Roughgarden, 2018]:
  - The Leximin allocation satisfies EFX + PO for agents with (general) identical valuations.
- Theorem [Plaut and Roughgarden, 2018]:
  - The Leximin allocation satisfies EFX + PO for two agents with (normalized) additive valuations.

Open Problem:
Does there always exist a complete allocation satisfying EFX?

# Satisfying EFX

- What about partial allocations satisfying EFX?
  - Easy! We can just throw all goods away and take the empty allocation.

- Theorem [Caragiannis et al. 2019]:
  - There exists a partial allocation that satisfies EFX and achieves a 2-approximation to the optimal Nash welfare.
  - No (complete or partial) EFX allocation can achieve a better approximation.

| | Existence | | Computation | |
|---|---|---|---|---|
| | Without PO | With PO | Without PO | With PO |
| Envy-Freeness | No | No | NP-hard | NP-hard |
| EFX | Open | Open | Open | Open |
| EF1 | Yes | Yes | Polytime | Open |
| Prop1 | Yes | Yes | Polytime | Polytime |

# Maximin Share

# Maximin Share [Budish 2011]

- "If I partition the goods into $n$ bundles and receive an adversarially chosen bundle, how much utility can I guarantee myself?"

- Define $MMS_i^k(S) = \max_{(P_1,\dots,P_k)\in\Pi_k(S)} \min_{1\le j\le k} v_i(P_j)$

- MMS allocation: One for which $v_i(A_i) \ge MMS_i^n(M)$

- Note that $MMS_i^n(M) \le \frac{v_i(M)}{n}$, so Proportionality implies MMS

# Maximin Share [Budish 2011]

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| $a_1$ | 2 | 1 | 3 | 0 | 1 | 2 |
| $a_2$ | 10 | 1 | 1 | 1 | 2 | 5 |
| $a_3$ | 3 | 1 | 3 | 0 | 5 | 2 |

$$MMS_1^n(M) = \min(3, 3, 3) = 3$$

$$MMS_2^n(M) = \min(10, 5, 5) = 5$$

$$MMS_3^n(M) = \min(4, 5, 5) = 4$$

# Achieving Maximin Allocations

- Theorem [Procaccia and Wang 2014]:
  - There exist instances for which no allocation satisfies MMS.

- Instead, consider approximations.
  - c-MMS: allocation for which $v_i(A_i) \geq c \cdot MMS_i^n(M)$
  - Guarantee $v_i(A_i) \geq MMS_i^k(M)$ for some $k > n$

- Theorem [Budish 2011]:
  - There always exists an allocation that satisfies $v_i(A_i) \geq MMS_i^{(n+1)}(M)$ for every agent $i$.

# c-MMS Allocations

- Theorem [Procaccia and Wang 2014]:
  - A (2/3)-MMS allocation always exists.
- Theorem [Amanatidis et al. 2017]:
  - A $(2/3\text{-}\epsilon)$-MMS allocation can be computed in polynomial time.
- Theorem [Ghodsi et al. 2018]:
  - A (3/4)-MMS allocation always exists and a $(3/4\text{-}\epsilon)$-MMS allocation can be computed in polynomial time.

# c-MMS Allocations

[Ghodsi et al. 2018]

| | Additive | Submodular | Subadditive |
|---|---|---|---|
| Lower bound (existence) | $\frac{3}{4}$ | $\frac{1}{3}$ | $\frac{1}{10\lceil \log m \rceil}$ |
| Lower bound (polynomial algorithm) | $\frac{3}{4} - \epsilon$ | $\frac{1}{3}$ | - |
| Upper bound | $1 - \frac{1}{n^n + 1}$ | $\frac{3}{4}$ | $\frac{1}{2}$ |

Open Problem:
Close the gaps!

# c-MMS Allocations for Bads

- Theorem [Aziz et al. 2017]:
  - A 2-MMS allocation always exists and can be computed in polynomial time when dividing bads.

- Theorem [Barman and Krishnamurthy 2017]:
  - A (4/3)-MMS allocation always exists and can be computed in polynomial time when dividing bads.

# Groupwise MMS [Barman et al. 2018]

- Idea: $MMS_i^k$ should be guaranteed for all groups $J$ of agents of size $k$ and set of goods $\cup_{i \in J} A_i$

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| $a_1$ | 5 | 5 | $5+\epsilon$ | $5-\epsilon$ | $5+\epsilon$ | $5-\epsilon$ |
| $a_2$ | 5 | 5 | $5+\epsilon$ | $5-\epsilon$ | $5+\epsilon$ | $5-\epsilon$ |
| $a_3$ | 10 | 10 | 0 | 0 | $\epsilon$ | $\epsilon$ |

- $v_3(A_3) \geq MMS_3^3(M)$ but $v_3(A_3) < MMS_3^2(A_1 \cup A_3)$

# Groupwise MMS [Barman et al. 2018]

- Allocation $A$ satisfies Groupwise Maximin Share (GMMS) if,

$$\forall i: v_i(A_i) \geq \max_{J \subseteq N} MMS_i^{|J|}(\cup_{j \in J} A_j)$$

- Theorem [Barman et al. 2018]:
  - When valuations are additive, a 0.5-GMMS allocation exists and can be found in polynomial time.
  - Algorithm: Select an agent who is not envied by any other agent, and allocate her her most preferred unallocated good.
  - Small refinement of EF1 algorithm from earlier

# (Relaxed) Equitability

# Equitability

- Recall equitability:

$$\forall i, j \in N: v_i(A_i) \geq v_j(A_j)$$

- We can relax it in the same way we did for envy-freeness [Gourves et al. 2014, Freeman et al. 2019].
- Equitability up to one good (EQ1):

$$\forall i, j \in N, \exists g \in A_j: v_i(A_i) \geq v_j(A_j \setminus \{g\})$$

- Equitability up to any good (EQX):

$$\forall i, j \in N, \forall g \in A_j: v_i(A_i) \geq v_j(A_j \setminus \{g\})$$

# Algorithm for Achieving EQX

- Greedy Algorithm [Gourves et al. 2014]:
  - Allocate to the lowest-utility agent the unallocated good that she values the most.

- Almost the same as EF1 algorithm, but achieves EQX!
  - Compare to EFX, existence still unknown

# EQ1/EQX + PO

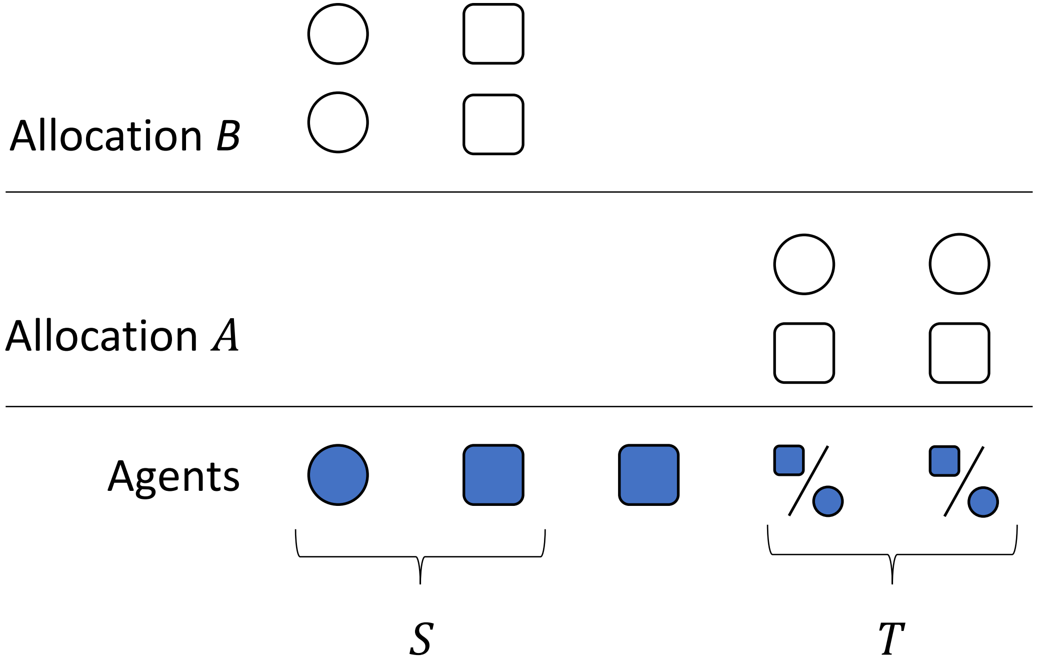
- Theorem [Freeman et al. 2019]:
  - An allocation satisfying EQ1 and PO may not exist.
  - Compare to EF1 + PO always exists

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| $a_1$ | 1 | 1 | 1 | 0 | 0 | 0 |
| $a_2$ | 0 | 0 | 0 | 1 | 1 | 1 |
| $a_3$ | 0 | 0 | 0 | 1 | 1 | 1 |

- Theorem [Freeman et al. 2019]:
  - When valuations are strictly positive, the Leximin allocation is EQX + PO

# Group Fairness

# Beyond Individual Fairness



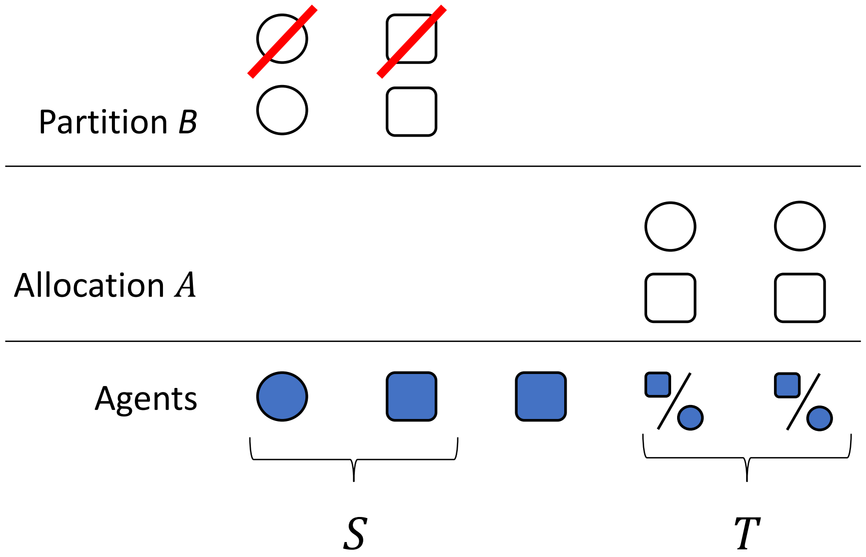

Envy-Free up to One Good (EF1)

# Group Fairness

- An allocation A is group fair if for every non-empty $S, T \subseteq N$ and every partition $(B_i)_{i \in S}$ of $\cup_{j \in T} A_j$, $\left(\frac{|S|}{|T|}\right) \cdot (v_i(B_i))_{i \in S}$ does not Pareto dominate $(v_i(A_i))_{i \in S}$
- “It should not be possible to redistribute the goods allocated to group T amongst group S in such a way that every member of group S is (weakly, with at least one strictly) better off, with utilities adjusted for group sizes”
- Group Fairness ⇒ EF + PO

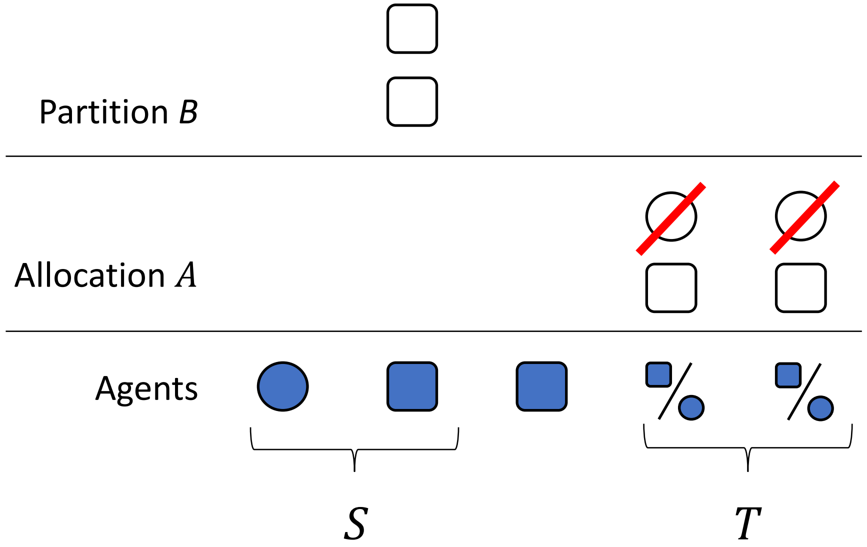
# Group Fairness Relaxations

- Group Fairness up to One Good, After (GF1A) [Conitzer et al. 2019]
  - "It should not be possible to redistribute the goods allocated to group T amongst group S in such a way that every member of group S is (weakly, with at least one strictly) better off, even when one good is removed from each agent in S, with utilities adjusted for group sizes"



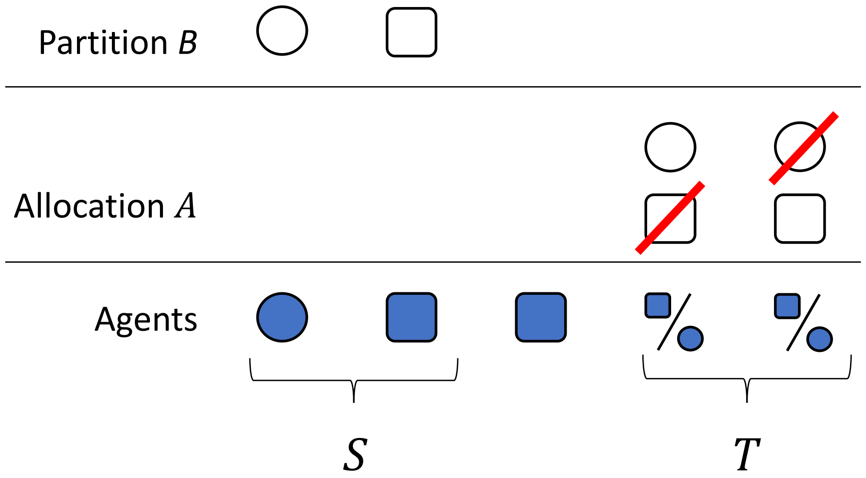
# Group Fairness Relaxations

- Group Fairness up to One Good, Before (GF1B) [Conitzer et al. 2019]
  - "It should not be possible to redistribute the goods allocated to group T, with one good per agent in T removed, amongst group S in such a way that every member of group S is (weakly, with at least one strictly) better off, with utilities adjusted for group sizes"



# Group Fairness Relaxations

- Group Fairness up to One Good, Before (GF1B) [Conitzer et al. 2019]
  - "It should not be possible to redistribute the goods allocated to group T, with one good per agent in T removed, amongst group S in such a way that every member of group S is (weakly, with at least one strictly) better off, with utilities adjusted for group sizes"



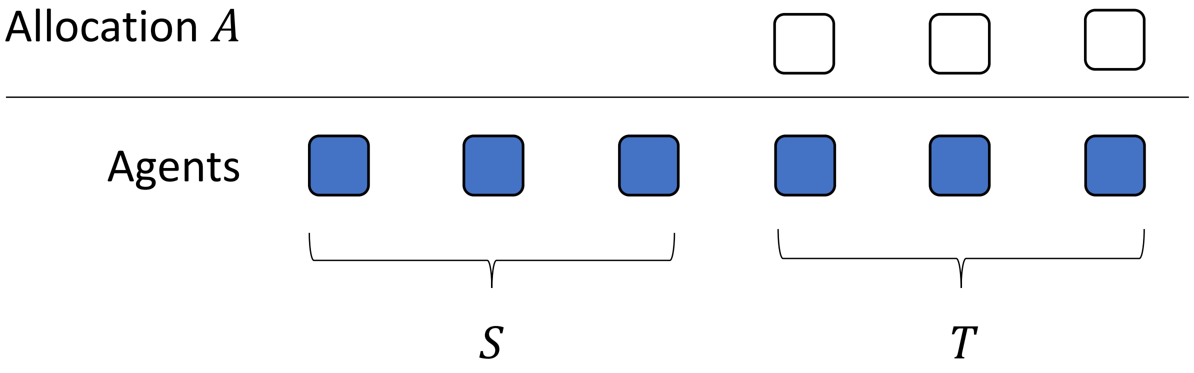
# Achieving GF1A/GF1B

- Locally Nash-optimal allocation: Product of utilities cannot be improved by moving a single good.
  $\forall i, j, g \in A_j$: $v_j(g) > 0$ and $v_i(A_i) \cdot v_j(A_j) \geq v_i(A_i + g) \cdot v_j(A_j - g)$
- Theorem [Conitzer et al. 2019]:
  - Any locally Nash-optimal allocation satisfies GF1A and GF1B.
  - Can be computed in pseudo-polynomial time by local search
  - When valuations are identical, an allocation is locally Nash-optimal iff it is EFX/EQX.

Open Problem:
Can we compute a locally Nash-optimal allocation in polynomial time?

# Known Groups

- When we want to provide guarantees for **all** subsets of agents, "up to one good per agent" guarantees are the best we can give.




Open Problem:

Can we give stronger guarantees when $S$ and $T$ are fixed in advance?

# Nash Welfare Approximation

# Nash Welfare Approximation

- We have seen that MNW satisfies several nice properties.
  - GF1A/B ($\Rightarrow$ EF1) + PO
  - Scale-free
  - Natural fairness/efficiency tradeoff
- But NP-hard to optimize. Can we approximate?
- Theorem [Lee 2017]
  - Computing an allocation that maximizes the geometric mean of agent utilities under additive valuation functions is APX-hard.
  - Approximating to within a factor of 1.00008 is NP-hard.

# Nash Welfare Approximation

- Theorem [Cole and Gkatzelis 2015, Cole et al 2017]:
  - There exists a polynomial time algorithm that approximates the MNW objective to within a constant multiplicative factor of 2.
- Theorem [Barman et al. 2018]:
  - There exists a polynomial time algorithm that approximates the MNW objective to within a constant multiplicative factor of 1.45.

Open Problem:

Close the gap between the 1.00008 lower bound and 1.45 upper bound.

# Nash Welfare Approximation

- Approximate MNW solutions may not retain the nice properties of the exact solution.

- Theorem [Garg and McGlaughlin 2019]:
  - There exists a polynomial time algorithm that approximates the MNW objective to within a constant multiplicative factor of 2 and achieves Prop1, (1/2n)-MMS and PO.

- And recall, there exists a partial allocation that satisfies EFX and is a 2-approximation to MNW objective [Caragiannis et al 2019].

# Price of Fairness

# Price of Fairness

- What effect does requiring a fairness property have on the social welfare?
- Price of Fairness [Bertsimas et al. 2011, Caragiannis et al. 2012]:
  - The price of fairness of fairness property P is defined as the ratio of the maximum possible social welfare and the maximum social welfare of an allocation that satisfies P.
- Strong Price of Fairness [Bei et al. 2019]:
  - The strong price of fairness of fairness property P is defined as the ratio of the maximum possible social welfare and the minimum social welfare of an allocation that satisfies P.
- Cf. Price of Stability and Price of Anarchy

# Price of Fairness

- Theorem [Caragiannis et al. 2012]:
  - The price of fairness for proportionality, envy-freeness and equitability are:

| | Indivisible Goods | Cake Cutting |
|---|---|---|
| Proportionality | $\Theta(n)$ | $\Theta(\sqrt{n})$ |
| Envy-freeness | $\Theta(n)$ | $\Theta(\sqrt{n})$ |
| Equitability | $\infty$ | $\Theta(n)$ |

- Caragiannis et al. also studied divisible items, and bads.

# Price of Fairness

- Theorem [Bei et al. 2019]:
  - Bounds on the (strong) price of fairness for indivisible goods

| | Price of P | Strong Price of P |
|---|---|---|
| EF1 | LB: $\Omega(\sqrt{n})$, UB: $O(n)$ | $\infty$ |
| Round Robin | $n$ | $n^2$ |
| Max Nash Welfare | $\Theta(n)$ | $\Theta(n)$ |
| Leximin | $\Theta(n)$ | $\Theta(n)$ |
| Pareto optimality | 1 | $\Theta(n^2)$ |

Open Problem: Close the gap for EF1

# Strategyproofness

# Adding Strategyproofness

- None of the rules we have considered so far are strategyproof
- For divisible goods, structure of strategyproof mechanisms is fairly rich
  - ➢ Impossibilities from the divisible realm carry over
- What about indivisible goods?

| | $g_1$ | $g_2$ | $g_3$ |
|---|---|---|---|
| $a_1$ | 1 | $x$ | 0 |
| $a_2$ | 0 | $y$ | 1 |

# Picking-Exchange Mechanisms

[Amanatidis et al. 2017]

- Picking Mechanism:
  - Partition $M = N_1 \cup N_2$
  - Agent 1 receives a subset of offers $O_1 \subseteq 2^{N_1}$. Let $S_1 = \arg\max_{S \in O_1} v_1(S)$.
  - Agent 2 receives a subset of offers $O_2 \subseteq 2^{N_2}$. Let $S_2 = \arg\max_{S \in O_2} v_2(S)$.
  - $A_1 = S_1 \cup (N_2 \setminus S_2)$ and $A_2 = S_2 \cup (N_1 \setminus S_1)$
- $N_1 = \{g_1, g_2, g_3, g_4\}, N_2 = \{g_5, g_6\}$
- $O_1 = \{\{g_1, g_2\}, \{g_2, g_3\}, \{g_4\}\}, O_2 = \{\{g_5\}, \{g_6\}\}$

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| $a_1$ | 3 | 5 | 5 | 10 | 4 | 2 |
| $a_2$ | 2 | 3 | 6 | 1 | 5 | 3 |

# Picking-Exchange Mechanisms

[Amanatidis et al. 2017]

- Exchange Mechanism:
  - Partition $M = E_1 \cup E_2$
  - Set of exchange deals $\mathrm{D} = \{(S_1, T_1), \dots, (S_k, T_k)\}$, where each $(S, T) \subseteq (E_1, E_2)$
  - Agent $i$ receives allocation $E_i$ by default, with exchanges performed if they are mutually beneficial
- $E_1 = \{g_1, g_2, g_3\}, E_2 = \{g_4, g_5\}$
- $D = \{(\{g_2, g_3\}, \{g_4\})\}$

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
|---|---|---|---|---|---|
| $a_1$ | 6 | 2 | 3 | 7 | 1 |
| $a_2$ | 1 | 6 | 1 | 4 | 7 |

# Picking-Exchange Mechanisms

[Amanatidis et al. 2017]

- Picking-Exchange Mechanism: Run a picking mechanism on $N_1 \cup N_2 \subseteq M$ and an exchange mechanism on $E_1 \cup E_2 \subseteq M$, where $N_1 \cup N_2 \cup E_1 \cup E_2 = M$ and $N_1, N_2, E_1, E_2$ are pairwise disjoint.
  - Up to tiebreaking technicalities…

# Picking-Exchange Mechanisms

[Amanatidis et al. 2017]

- Theorem [Amanatidis et al. 2017]:
  - For $n = 2$ an allocation mechanism that allocates all goods is strategyproof if and only if it is a picking-exchange mechanism
- Corollary [Amanatidis et al. 2017]:
  - For $n = 2$, any strategyproof mechanism that allocates all goods does not achieve any positive approximation of the minimum envy or best proportionality guarantee.
  - For $n = 2$ and $m \geq 5$, no strategyproof mechanism can allocate all items and satisfy EF1.
  - For $n = 2$, no strategyproof mechanism guarantees better than $\frac{1}{\lfloor m/2 \rfloor}$-MMS.
    - This is a tight bound [Amanatidis et al. 2016]

# More General Strategyproof Mechanisms

Open Problem:
What is the structure of strategyproof mechanisms for $n = 2$ when not all goods have to be allocated?

Open Problem:
What is the structure of strategyproof mechanisms for $n > 2$?

# What's Not Covered

- Envy-freeness up to one less-preferred item (EFL) [Barman et al. 2018]
  - Stronger than EF1 and guaranteed to exist
  - Existence of EFL + PO allocations is an open question
- Various constraints and additional features
  - Agent social network structure
  - Connectivity constraints when items lie on a graph
- Asymptotic results
- …

# Ordinal Preferences

# Ordinal Preferences

- Instead of valuation functions, take in preference orderings $\succcurlyeq_i$ over items
  - E.g. $g_2 \succcurlyeq_i g_3 \succcurlyeq_i g_1 \succcurlyeq_i g_4$

- Agents are assigned fractions of each item
  - $A = \left(A_{i,j}\right)_{i\in[n],j\in[m]}$
  - Can be interpreted as lotteries over integral allocations

# Ordinal Preferences

- Partial preferences over bundles defined via stochastic dominance extension

$$A \succcurlyeq_i^{SD} B \quad \text{iff} \qquad \forall k: \sum_{j \succcurlyeq_i k} A_{i,j} \geq \sum_{j \succcurlyeq_i k} B_{i,j}$$

- Many other extensions possible
  - Upper/downward lexicographic [Cho 2012]
  - Pairwise comparison [Aziz et al. 2014]
  - Bilinear dominance [Aziz et al. 2014]
- Can also elicit ordinal information over subsets directly [Bouveret et al. 2010]

# Two Mechanisms

- Random Priority
  - Select a random ordering of the agents. Agents select their favorite $m/n$ goods in order.
- Probabilistic Serial [Bogomolnaia and Moulin 2001]
  - Agents "eat" at a constant (equal) rate. At any time, agents eat their most preferred good that is not completely consumed.
- $\succcurlyeq_1: g_1 \succcurlyeq_1 g_2 \succcurlyeq_1 g_3 \succcurlyeq_1 g_4$ $\quad$ $\succcurlyeq_2: g_2 \succcurlyeq_2 g_3 \succcurlyeq_2 g_1 \succcurlyeq_2 g_4$

Random Priority

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ |
|---|---|---|---|---|
| $a_1$ | 1 | 1/2 | 0 | 1/2 |
| $a_2$ | 0 | 1/2 | 1 | 1/2 |

Probabilistic Serial

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ |
|---|---|---|---|---|
| $a_1$ | 1 | 0 | 1/2 | 1/2 |
| $a_2$ | 0 | 1 | 1/2 | 1/2 |

# SD-efficiency

- SD-efficiency: There should not exist an alternative allocation that all agents weakly prefer and some agent strictly prefers.
- Theorem [Bogomolnaia and Moulin 2001]:
  - Probabilistic Serial satisfies SD-efficiency
- Random Priority is not SD-efficient
  - $\succcurlyeq_1: g_1 \succcurlyeq_1 g_2 \succcurlyeq_1 g_3 \succcurlyeq_1 g_4$ $\quad$ $\succcurlyeq_2: g_2 \succcurlyeq_2 g_1 \succcurlyeq_2 g_4 \succcurlyeq_2 g_3$

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ |
|---|---|---|---|---|
| $a_1$ | 1/2 | 1/2 | 1/2 | 1/2 |
| $a_2$ | 1/2 | 1/2 | 1/2 | 1/2 |

# SD-strategyproofness

- SD-strategyproofness: No agent should be able to improve their allocation by misreporting their preferences.
- Theorem:
  - Random Priority is SD-strategyproof.
- Probabilistic Serial is not SD-strategyproof
  - $\succcurlyeq_1$: ~~$g_1 \succcurlyeq_1 g_2$~~ $\succcurlyeq_1 g_3 \succcurlyeq_1 g_4$ $\quad$ $\succcurlyeq_2$: $g_2 \succcurlyeq_2 g_3 \succcurlyeq_2 g_1 \succcurlyeq_2 g_4$

    $g_2 \succcurlyeq_1 g_1$

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ |
|---|---|---|---|---|
| $a_1$ | 1 | 0 | 1/2 | 1/2 |
| $a_2$ | 0 | 1 | 1/2 | 1/2 |

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ |
|---|---|---|---|---|
| $a_1$ | 1 | 1/2 | 0 | 1/2 |
| $a_2$ | 0 | 1/2 | 1 | 1/2 |

# SD-Efficiency + SD-Strategyproofness

- Theorem [Bogomolnaia and Moulin 2001]:
  - No mechanism satisfies SD-efficiency, SD-strategyproofness, and equal treatment of equals

- We can get SD-efficiency + SD-envy-freeness
  - SD-envy-freeness: $\forall i, j$: $\sum_{j=1}^{m} A_{i,j} g_j \succcurlyeq_i^{SD} \sum_{j=1}^{m} B_{i,j} g_j$
  - Probabilistic Serial is SD-envyfree

# Public Decisions

# Public Decisions Model

- Set of agents $N$
- Set of issues $T$
- Each issue has associated set of alternatives $C^t = \{c_1^t, \dots, c_{k_t}^t\}$
- Agents have utility functions $u_i^t : A^t \to \mathbb{R}_+$

| | Issue 1 | | | Issue 2 | | | | Issue T | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $c_1^1$ | $c_2^1$ | $c_3^1$ | $c_1^2$ | $c_2^2$ | $c_3^2$ | | $c_1^T$ | $c_2^T$ | $c_3^T$ |
| $a_1$ | 3 | 1 | 0 | 2 | 5 | 1 | … | 6 | 5 | 5 |
| $a_2$ | 2 | 2 | 1 | 3 | 4 | 1 | | 2 | 4 | 3 |
| $a_3$ | 0 | 0 | 4 | 4 | 3 | 2 | | 5 | 4 | 5 |

# Public Decisions Model

- Set of agents $N$
- Set of issues $T$
- Each issue has associated set of alternatives $C^t = \{c_1^t, \ldots, c_{k_t}^t\}$
- Agents have utility functions $u_i^t: A^t \rightarrow \mathbb{R}_+$

| | Monday | | | Tuesday | | | | Sunday | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | |
| $a_1$ | 3 | 1 | 0 | 2 | 5 | 1 | … | 6 | 5 | 5 |
| $a_2$ | 2 | 2 | 1 | 3 | 4 | 1 | | 2 | 4 | 3 |
| $a_3$ | 0 | 0 | 4 | 4 | 3 | 2 | | 5 | 4 | 5 |

# Item Allocation as a Special Case

- Define the set of issues $T = M = \{g_1, \dots, g_m\}$
- Alternatives $C^t = N = \{a_1, \dots, a_n\}$
- $u_i^t(a_j) = \begin{cases} v_i(g_t) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

| | $g_1$ | $g_2$ | $g_3$ |
|---|---|---|---|
| $a_1$ | 5 | 2 | 3 |
| $a_2$ | 0 | 3 | 1 |
| $a_3$ | 2 | 3 | 4 |

| | $g_1$ | | | $g_2$ | | | $g_3$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $a_1$ | $a_2$ | $a_3$ | $a_1$ | $a_2$ | $a_3$ | $a_1$ | $a_2$ | $a_3$ |
| $a_1$ | 5 | 0 | 0 | 2 | 0 | 0 | 3 | 0 | 0 |
| $a_2$ | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 1 | 0 |
| $a_3$ | 0 | 0 | 2 | 0 | 0 | 3 | 0 | 0 | 4 |

# Fairness for Public Decisions

- Envy-freeness (and relaxations) not sensible in the general case
  - Decisions are public, all agents receive the same outcome
- Proportionality is still sensible
  - Each agent should receive their "dictator utility" multiplied by $1/n$
- Proportionality up to one *issue* (Prop1)
  - Each agent would receive their proportional share if they were allowed to change the outcome on a single issue
- Theorem [Conitzer et al. 2017]:
  - The MNW outcome satisfies Prop1 + PO in the public decisions setting
- Other fairness desiderata ((approximate) core, round robin share,…)

# Allocation of Public Goods

| | Issue 1 | | | Issue 2 | | |
|---|---|---|---|---|---|---|
| | $c_1^1$ | $c_2^1$ | $c_3^1$ | $c_1^2$ | $c_2^2$ | $c_3^2$ |
| $a_1$ | 3 | 1 | 0 | 2 | 5 | 1 |
| $a_2$ | 2 | 2 | 1 | 3 | 4 | 1 |
| $a_3$ | 0 | 0 | 4 | 4 | 3 | 2 |

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| $a_1$ | 3 | 1 | 0 | 2 | 5 | 1 |
| $a_2$ | 2 | 2 | 1 | 3 | 4 | 1 |
| $a_3$ | 0 | 0 | 4 | 4 | 3 | 2 |

- Generalizes public decisions
- A set of public goods $\{g_1, \ldots, g_m\}$
  - ➢ Each good can give a positive utility to multiple agents simultaneously
- Constraints on which subsets of public goods are feasible

# Allocation of Public Goods

| | Issue 1 | | | Issue 2 | | |
|---|---|---|---|---|---|---|
| | $c_1^1$ | $c_2^1$ | $c_3^1$ | $c_1^2$ | $c_2^2$ | $c_3^2$ |
| $a_1$ | 3 | 1 | 0 | 2 | 5 | 1 |
| $a_2$ | 2 | 2 | 1 | 3 | 4 | 1 |
| $a_3$ | 0 | 0 | 4 | 4 | 3 | 2 |

| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ |
|---|---|---|---|---|---|---|
| $a_1$ | 3 | 1 | 0 | 2 | 5 | 1 |
| $a_2$ | 2 | 2 | 1 | 3 | 4 | 1 |
| $a_3$ | 0 | 0 | 4 | 4 | 3 | 2 |

- Public decision example:
  - Exactly one of $\{g_1, g_2, g_3\}$ and exactly one of $\{g_4, g_5, g_6\}$ must be chosen
  - Partition matroid constraint

# Fairness Guarantees

- $(\delta, \alpha)$-Core
  - An allocation of public goods $C$ is in core if for every subset of agents $S \subseteq N$, there is no feasible allocation of public goods $C'$ such that
    $$\frac{|S|}{n} \cdot u_i(C') \geq u_i(C)$$
    for all $i \in S$, and at least one inequality is strict.

- Valuations are normalized so that $\max_j u_i(g_j) = 1$
- Core (i.e. $(0,0)$-core) generalizes proportionality
  - $(0,1)$-core generalizes a guarantee very similar to Prop1

# Fair Allocation of Public Goods

- Matroid constraints
  - Public goods are ground set elements
  - Feasible allocations are basis of a matroid
  - Generalizes public decisions (thus goods allocation) and multiwinner voting
- Theorem [Fain et al. 2018]
  - For matroid constraints, a $(0,2)$-core allocation exists, and for constant $\epsilon > 0$, a $(0,2+\epsilon)$-core allocation can be computed in polynomial time.
  - Algorithm: Maximize smooth Nash welfare $\prod_{i \in N}\left(1 + u_i(C)\right)$
  - For $\epsilon > 0$, $(0,1-\epsilon)$-core allocations may not exist.

Open Problem: Does there always exist a $(0,1)$-core allocation?

# Fair Allocation of Public Goods

- Theorem [Fain et al. 2018]
  - For "matching constraints" and constant $\delta \in (0,1]$, a $(\delta, 8 + 6/\delta)$-core allocation can be computed in polynomial time.
  - Algorithm: Maximize a slightly different smooth NW $\prod_{i \in N} \left(1 + 4/\delta + u_i(C)\right)$
  - For $\delta > 0$ and $\alpha < 1$, a $(\delta, \alpha)$-core allocation may not exist.
  - Open problem: Does there always exist a $(0,1)$-core allocation?

- A slightly worse guarantee with logarithmically large $\alpha$ in case of "packing constraints"

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