Recent Advances in Fair Resource Allocation

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Disclaimer

• In this tutorial, we will NOT
  ➢ Assume any prior knowledge of fair division
  ➢ Walk you through tedious, detailed proofs
  ➢ Claim to present a complete overview of the entire fair division realm
  ➢ Present (recent) unpublished results

• Instead, we will
  ➢ Focus mostly on the case of “additive preferences” for coherence
    o With some results for and pointers to domains with non-additive preferences

• If you spot any errors, missing results, or incorrect attributions:
  ➢ Please email nisarg@cs.toronto.edu or Rupert.Freeman@microsoft.com
Outline

• Fairness Axioms
  ➢ Proportionality
  ➢ Envy-freeness
  ➢ Maximin share guarantee
  ➢ Groupwise fairness
    o Core
    o Group envy-freeness
    o Groupwise MMS
    o Group fairness

• Implications of fairness
  ➢ Price of fairness
  ➢ Interplay with strategyproofness and Pareto optimality
  ➢ Restricted cases

• Settings
  ➢ Cake-cutting
  ➢ Homogeneous divisible goods
  ➢ Indivisible goods
A Generic Resource Allocation Framework

• A set of agents $N = \{1, 2, \ldots, n\}$

• A set of resources $M$
  ➢ May be finite or infinite

• Valuations
  ➢ Valuation of agent $i$ is $v_i : 2^M \rightarrow \mathbb{R}$
  ➢ Range is $\mathbb{R}_+$ when resources are *goods*, and $\mathbb{R}_-$ when they are *bads*

• Allocations
  ➢ $A = (A_1, \ldots, A_n) \in \Pi_n(M)$ is a partition of resources among agents
    o $A_i \cap A_j = \emptyset, \forall i, j \in N$ and $\bigcup_{i \in N} A_i = M$
  ➢ A partial allocation $A$ may have $\bigcup_{i \in N} A_i \neq M$
Cake Cutting

• Formally introduced by Steinhaus [1948]
• Agents: \( N = \{1, 2, \ldots, n\} \)
• Resource (cake): \( M = [0, 1] \)
• Constraints on an allocation \( A \)
  ➢ The entire cake is allocated (full allocation)
  ➢ Each \( A_i \in \mathcal{A} \), where \( \mathcal{A} \) is the set of finite unions of disjoint intervals
• **Simple allocations**
  ➢ Each agent is allocated a single interval
  ➢ Cuts cake at \( n - 1 \) points
Agent Valuations

• Each agent $i$ has an integrable density function $f_i: [0,1] \rightarrow \mathbb{R}_+$

• For each $X \in \mathcal{A}$, $v_i(X) = \int_{x \in X} f_i(x) dx$

• For normalization, we require $\int_0^1 f_i(x) dx = 1$
  ➢ Without loss of generality
Agent Valuations

• In this model, the valuations satisfy the following properties

• **Normalized**: $v_i([0,1]) = 1$

• **Divisible**: $\forall \lambda \in [0,1]$ and $I = [x, y]$, $\exists z \in [x, y]$ s.t. $v_i([x, z]) = \lambda v_i([x, y])$

• **Additive**: For disjoint intervals $I$ and $I'$, $v_i(I) + v_i(I') = v_i(I \cup I')$
Complexity

• Inputs are functions
  - Infinitely many bits may be needed to fully represent the input
  - Query complexity is more useful

• Robertson-Webb Model
  - Eval\(_i\)(x, y) returns \(v_i([x, y])\)
  - Cut\(_i\)(x, \(\alpha\)) returns y such that \(v_i([x, y]) = \alpha\)
Three Classic Fairness Desiderata

• Proportionality (Prop): \( \forall i \in N: v_i(A_i) \geq \frac{1}{n} \)
  ➢ Each agent should receive her “fair share” of the utility.

• Envy-Freeness (EF): \( \forall i, j \in N: v_i(A_i) \geq v_i(A_j) \)
  ➢ No agent should wish to swap her allocation with another agent.

• Equitability (EQ): \( \forall i, j \in N : v_i(A_i) = v_j(A_j) \)
  ➢ All agents should have the exact same value for their allocations.
  ➢ No agent should be jealous of what another agent received.
Example

- Value density functions

- Agent 1 wants $[0, \frac{1}{3}]$ uniformly and does not want anything else

- Agent 2 wants the entire cake uniformly

- Agent 3 wants $[\frac{2}{3}, 1]$ uniformly and does not want anything else
Example

• Value density functions

• Consider the following allocation

- $A_1 = [0, \frac{1}{9}] \Rightarrow v_1(A_1) = \frac{1}{3}$
- $A_2 = [\frac{1}{9}, \frac{8}{9}] \Rightarrow v_2(A_2) = \frac{7}{9}$
- $A_3 = [\frac{8}{9}, 1] \Rightarrow v_3(A_3) = \frac{1}{3}$

• The allocation is proportional, but not envy-free or equitable
Example

- Consider the following allocation

- $A_1 = [0, \frac{1}{6}] \Rightarrow v_1(A_1) = \frac{1}{2}$
- $A_2 = [\frac{1}{6}, \frac{5}{6}] \Rightarrow v_2(A_2) = \frac{2}{3}$
- $A_3 = [\frac{5}{6}, 1] \Rightarrow v_3(A_3) = \frac{1}{2}$

- The allocation is proportional and envy-free, but not equitable

- Value density functions
Example

- Consider the following allocation
  - $A_1 = [0, 1/5] \Rightarrow v_1(A_1) = 3/5$
  - $A_2 = [1/5, 4/5] \Rightarrow v_2(A_2) = 3/5$
  - $A_3 = [4/5, 1] \Rightarrow v_3(A_3) = 3/5$

- The allocation is proportional, envy-free, and equitable
Relations Between Fairness Desiderata

• Envy-freeness implies proportionality
  ➢ Summing $v_i(A_i) \geq v_i(A_j)$ over all $j$ gives proportionality

• For 2 agents, proportionality also implies envy-freeness
  ➢ Hence, they are equivalent.

• Equitability is incomparable to proportionality and envy-freeness
  ➢ E.g. if each agent has value 0 for her own allocation and 1 for the other agent’s allocation, it is equitable but not proportional or envy-free.
Existence

• Theorem [Alon, 1987]
  Suppose the value density function $f_i$ of each agent valuation $v_i$ is continuous. Then, we can cut the cake at $n^2 - n$ places and rearrange the $n^2 - n + 1$ intervals into $n$ pieces $A_1, ..., A_n$ such that
  \[ v_i(A_j) = \frac{1}{n}, \forall i, j \in N \]

• This is called a “perfect partition”
  ➢ It is trivially envy-free (thus proportional) and equitable

• As we will later see, this cannot be found with finitely many queries in Robertson-Webb model
Proportionality
PROPORTIONALITY : $n = 2$ AGENTS

• **CUT-AND-CHOOSE**
  - Agent 1 cuts the cake at $x$ such that $v_1([0, x]) = v_1([x, 1]) = 1/2$
  - Agent 2 chooses the piece that she prefers.

• Elegant protocol
  - Proportional (equivalent to envy-freeness for 2 agents)
  - Needs only one cut and one eval query (optimal)

• More agents?
PROPORTIONALITY: DUBINS-SPANIER

• **DUBINS-SPANIER**
  - Referee starts a knife at 0 and moves the knife to the right.
  - Repeat: When the piece to the left of the knife is worth $\frac{1}{n}$ to an agent, the agent shouts “stop”, receives the piece, and exits.
  - When only one agent remains, she gets the remaining piece.

• Can be implemented easily in Robertson-Webb model
  - When $[x, 1]$ is left, ask each remaining agent $i$ to cut at $y_i$ so that $v_i([x, y_i]) = \frac{1}{n}$, and give agent $i^* \in \arg \min_i y_i$ the piece $[x, y_i^*]$.

• Query complexity: $\Theta(n^2)$
PROPORTIONALITY: EVEN-PAZ

• EVEN-PAZ

• Input:
  ➢ Interval $[x, y]$, number of agents $n$ (assume a power of 2 for simplicity)

• Recursive procedure:
  ➢ If $n = 1$, give $[x, y]$ to the single agent.
  ➢ Otherwise:
    o Each agent $i$ marks $z_i$ such that $v_i([x, z_i]) = v_i([z_i, y])$
    o $z^* = (n/2)^{th}$ mark from the left.
    o Recurse on $[x, z^*]$ with the left $n/2$ agents, and on $[z^*, y]$ with the right $n/2$ agents.

• Query complexity: $\Theta(n \log n)$
Complexity of Proportionality

• **Theorem [Edmonds and Pruhs, 2006]:**
  - Any protocol returning a proportional allocation needs $\Omega(n \log n)$ queries in the Robertson-Webb model.

• Hence, **EVEN-PAZ** is provably (asymptotically) optimal!
Envy-Freeness
Envy-Freeness : Few Agents

• $n = 2$ agents : **CUT-AND-CHOOSE** (2 queries)
• $n = 3$ agents : **SELFridge-CONWAY** (14 queries)

Suppose we have three players $P_1$, $P_2$ and $P_3$. Where the procedure gives a criterion for a decision it means that criterion gives an optimum choice for the player.

1. $P_1$ divides the cake into three pieces he considers of equal size.
2. Let’s call $A$ the largest piece according to $P_2$.
3. $P_2$ cuts off a bit of $A$ to make it the same size as the second largest. Now $A$ is divided into: the trimmed piece $A_1$ and the trimmings $A_2$. Leave the trimmings $A_2$ to the side for now.
   - If $P_2$ thinks that the two largest parts are equal (such that no trimming is needed), then each player chooses a part in this order: $P_3$, $P_2$ and finally $P_1$.
4. $P_3$ chooses a piece among $A_1$ and the two other pieces.
5. $P_2$ chooses a piece with the limitation that if $P_3$ didn’t choose $A_1$, $P_2$ must choose it.
6. $P_1$ chooses the last piece leaving just the trimmings $A_2$ to be divided.

It remains to divide the trimmings $A_2$. The trimmed piece $A_1$ has been chosen by either $P_2$ or $P_3$, let’s call the player who chose it $P_A$ and the other player $P_B$.

1. $P_B$ cuts $A_2$ into three equal pieces.
2. $P_A$ chooses a piece of $A_2$ - we name it $A_{21}$.
3. $P_1$ chooses a piece of $A_2$ - we name it $A_{22}$.
4. $P_B$ chooses the last remaining piece of $A_2$ - we name it $A_{23}$. Gets complex pretty quickly!
Envy-Freeness : Few Agents

• [Brams and Taylor, 1995]
  ➢ The first finite (but unbounded) protocol for any number of agents

• [Aziz and Mackenzie, 2016a]
  ➢ The first bounded protocol for 4 agents (at most 203 queries)

• [Amanatidis et al., 2018]
  ➢ A simplified version of the above protocol for 4 agents (at most 171 queries)
Envy-Freeness

• Theorem [Aziz and Mackenzie, 2016b]
  ➢ There exists a bounded protocol for computing an envy-free allocation with \( n \) agents, which requires \( O(n^{n^{nnn}}) \) queries
  ➢ After \( O(n^{2n+3}) \) queries, the protocol can output a partial allocation that is both proportional and envy-free

• What about lower bounds?
Complexity of Envy-Freeness

• Theorem [Procaccia, 2009]
  Any protocol for finding an envy-free allocation requires $\Omega(n^2)$ queries.

Open Problem

Bridge the gap between $O(n^{nnnn})$ upper bound and $\Omega(n^2)$ lower bound for envy-free cake-cutting

• Theorem [Stromquist, 2008]
  There is no finite (even unbounded) protocol for finding a simple envy-free allocation for $n \geq 3$ agents.
Equitability
Upper Bound: $n = 2$ Agents

• Existence
  ➢ Suppose we cut the cake at $x$ to form pieces $[0, x]$ and $[x, 1]$
  ➢ Let $f(x) = v_1([0, x]) - v_2([x, 1])$
    o Note that $f(0) = -1$, $f(1) = 1$, and $f$ is continuous
  ➢ By the intermediate value theorem: $\exists x^*$ such that $f(x^*) = 0$
  ➢ Allocation $A_1 = [0, x^*]$ and $A_2 = [x^*, 1]$ is equitable

• Theorem [Cechlárová and Pillárová, 2012]
  ➢ Using binary search for $x^*$, we can find an $\epsilon$-equitable allocation for 2 agents
    with $O(\ln(1/\epsilon))$ queries.
Upper Bound: $n > 2$ Agents

• Theorem [Cechlárová and Pillárová, 2012]
  - This technique can be extended to $n$ agents to find an $\epsilon$-equitable allocation in $O(n \ln(1/\epsilon))$ queries.

• Theorem [Procaccia and Wang, 2017]
  - There exists a protocol for $n$ agents which finds an $\epsilon$-equitable allocation in $O(1/\epsilon \ln(1/\epsilon))$ queries.
  - Intuition:
    - If $n \leq 1/\epsilon$, use above protocol for finding an equitable $\epsilon$-equitable allocation.
    - If $n > 1/\epsilon$, use a variant of the Evan-Paz algorithm to find an anti-proportional allocation where $n' = \lceil 1/\epsilon \rceil$ agents get value at most $1/n'$, and the rest receive nothing.
      - While this is a “bad” allocation, it is $\epsilon$-equitable.
Lower Bound

• **Theorem [Procaccia and Wang, 2017]**
  Any protocol for finding an $\epsilon$-equitable allocation must require $\Omega\left(\frac{\ln(1/\epsilon)}{\ln \ln(1/\epsilon)}\right)$ queries.

• **Theorem [Procaccia and Wang, 2017]**
  There is no finite (even if unbounded) protocol for finding an equitable allocation.
  - Non-existence of bounded protocols follows from the previous result.
  - But their proof works for non-existence of unbounded protocols as well.
Price of Fairness
Price of Fairness

• Measures the worst-case loss in social welfare due to requirement of a fairness property $X$

• **Social welfare** of allocation $A$ is the sum of values of the agents
  ➢ Denoted $sw(A) = \sum_{i \in N} v_i(A_i)$

• Let $\mathcal{F}$ denote the set of feasible allocations and $\mathcal{F}_X$ denote the set of feasible allocations satisfying property $X$

$$PoF_X = \sup_{v_1, \ldots, v_n} \frac{\max_A sw(A)}{\max_{A \in \mathcal{F}_X} sw(A)}$$
Price of Fairness

• Theorem [Caragiannis et al., 2009]
  For cake-cutting, the price of proportionality is $\Theta(\sqrt{n})$, and the price of equitability is $\Theta(n)$.

• Theorem [Bertsimas et al., 2011]
  For cake-cutting, the price of envy-freeness is also $\Theta(\sqrt{n})$. This is achieved by an allocation maximizing the Nash welfare $\Pi_i v_i(A_i)$.

  Fun fact: The price of EF in cake-cutting was mentioned as an open question in a previous version of this tutorial, and was also believed to be open by many groups of researchers until recently.
Efficiency
Efficiency

• Weak Pareto optimality (WPO)
  - Allocation $A$ is weakly Pareto optimal if there is no allocation $B$ such that $v_i(B_i) > v_i(A_i)$ for all $i \in N$.
  - “Can’t make everyone happier”

• Pareto optimality (PO)
  - Allocation $A$ is Pareto optimal if there is no allocation $B$ such that $v_i(B_i) \geq v_i(A_i)$ for all agents $i \in N$, and at least one inequality is strict.
  - “Can’t make someone happier without making someone else less happy”
  - Easy to achieve in isolation (e.g. “serial dictatorship”)
PO+EF+EQ: (Non-)Existence

• Theorem [Barbanel and Brams, 2011]
  With two agents, there always exists an allocation that is envy-free (thus proportional), equitable, and Pareto optimal.
  ➢ Their algorithm has similarities to the more popular “adjusted winner” algorithm, which we will see later in the tutorial.

• With $n \geq 3$ agents, PO+EQ is impossible
What about PO+EF?

• **Competitive Equilibrium from Equal Incomes (CEEI)**
  ➢ At equilibrium: there is an additive price function $P$ on the cake, and each agent gets to buy their best piece from a budget of one unit of fake currency
  
  ➢ **WCE:** $\forall i \in N, Z \subseteq [0,1]: P(Z) \leq P(A_i) \Rightarrow v_i(Z) \leq v_i(A_i)$
  ➢ **EI:** $\forall i \in N: P(A_i) = 1$

• **Theorem [Weller, 1985]**
  For cake-cutting, a CEEI always exists. Every CEEI is both envy-free and weakly Pareto optimal.
s-CEEI

• Strong Competitive Equilibrium from Equal Incomes (s-CEEI)
  ➢ A positive slice $Z$ is a subset of the cake valued positively by at least one agent
  ➢ Allocation $A$ is called s-CEEI allocation if there exists an additive price function $P$ satisfying
    
    $P(Z) > 0$ iff $Z$ is a positive slice
    
    SCE: $\forall i \in N$, and positive slices $Z \subseteq [0,1]$ and $Z_i \subseteq A_i$: $\frac{v_i(Z_i)}{P(Z_i)} \geq \frac{v_i(Z)}{P(Z)}$
    
    EI: $\forall i \in N: P(A_i) = 1$

• Theorem [Segal-Halevi and Sziklai, 2018]
  For cake-cutting, an s-CEEI allocation always exists. Every s-CEEI allocation is envy-free and Pareto optimal.
s-CEEI and Nash-Optimality

• An allocation $A^*$ is called Nash-optimal if

$$A^* \in \arg \max_A \Pi_{i \in N} v_i(A_i)$$

• Theorem [Segal-Halevi and Sziklai, 2018]
  For cake-cutting, the set of s-CEEI allocations is exactly the same as the set of Nash-optimal allocations.
Due to PO, suppose:

- Agent 1 gets $x$ fraction of $[0, \frac{2}{3}]$
- Agent 2 gets $1 - x$ fraction of $[0, \frac{2}{3}]$ and all of $[\frac{2}{3}, 1]$
- $v_1(A_1) = x$
- $v_2(A_2) = (1 - x) \cdot \frac{2}{3} + \frac{1}{3} = \frac{(3-2x)}{3}$

Maximize $x \cdot \frac{(3-2x)}{3} \Rightarrow x = \frac{3}{4}$

Nash-optimal allocation:
- $A_1 = [0, \frac{1}{2}]$, $v_1(A_1) = \frac{3}{4}$
- $A_2 = [\frac{1}{2}, 1]$, $v_2(A_2) = \frac{1}{2}$
Nash-Optimality = s-CEEI

• Still must be PO, so like before
  ➢ Agent 1 buys $x$ fraction of $[0, \frac{2}{3}]$
  ➢ Agent 2 buys $1 - x$ fraction of $[0, \frac{2}{3}]$ and all of $[\frac{2}{3}, 1]$

• Prices: $P([0, \frac{2}{3}]) = a, P([\frac{2}{3}, 1]) = b$
  ➢ Spending: $a \cdot x = 1, a \cdot (1 - x) + b = 1$
    ○ Hence, $a + b = 2$

• Two cases: $x < 1$ or $x = 1$
Nash-Optimality = s-CEEI

- $x < 1$
  - Agent 2 buys parts of both pieces
  - MBB:
    \[
    \frac{1/3}{b} = \frac{2/3}{a} \Rightarrow a = 2b \Rightarrow (a, b) = (4/3, 2/3)
    \]
  - Substituting in $a \cdot x = 1$, we get $x = 3/4$
    - Same as Nash-optimal solution
Nash-Optimality = s-CEEI

• $x = 1$
  ➢ Since $a \cdot x = 1, a \cdot (1 - x) + b = 1$, we get that $a = b = 1$
  ➢ Agent 2 buys the second piece, so by MBB:
    \[
    \frac{1/3}{b} \geq \frac{2/3}{a} \Rightarrow a \geq 2b
    \]
  ➢ Contradiction!
  ➢ So there is no s-CEEI with $x = 1$
Strategyproofness
Strategyproofness (SP)

• Direct-revelation mechanisms
  ➢ A direct-revelation mechanism $h$ takes as input all the valuation functions $v_1, \ldots, v_n$, and returns an allocation $A$
  ➢ Notation: $h(v_1, \ldots, v_n) = A$, $h_i(v_1, \ldots, v_n) = A_i$

• Strategyproofness (deterministic mechanisms)
  ➢ A direct-revelation mechanism $h$ is called strategyproof if
    \[
    \forall v_1, \ldots, v_n, \forall i, \forall v'_i : v_i(h_i(v_1, \ldots, v_n)) \geq v_i(h_i(v_1, \ldots, v'_i, \ldots, v_n))
    \]
  ➢ That is, no agent $i$ can achieve a higher value by misreporting her valuation, regardless of what the other agents report
Strategyproofness (SP)

• **Strategyproofness (randomized mechanisms)**
  - Technically, referred to as “truthfulness-in-expectation”
    - When referring to SP for randomized mechanisms, we will refer to this concept

  - A randomized direct-revelation mechanism \( h \) is called strategyproof if
    \[
    \forall v_1, ..., v_n, \forall i, \forall v'_i : E\left[v_i(h_i(v_1, ..., v_n))\right] \geq E\left[v_i(h_i(v_1, ..., v'_i, ..., v_n))\right]
    \]

  - That is, no agent \( i \) can achieve a higher *expected* value by misreporting her valuation, regardless of what the other agents report
    - Expectation is over the randomness of the mechanism
Deterministic SP Mechanisms

• Theorem [Menon and Larson ’17, Bei et al. ‘17]
  No non-wasteful deterministic SP mechanism is (even approximately) proportional.
  ➢ Since EF is at least as strict as Prop, SP+EF is also impossible subject to non-wastefulness.
  ➢ Non-wastefulness can be replaced by a requirement of “connected pieces”, and the impossibility result still holds.

Open Problem
Does the SP+Prop impossibility hold even without the non-wastefulness assumption?
Deterministic SP Mechanisms

• SP+PO is easy to achieve
  ➢ E.g. serial dictatorship

• SP+PO+EQ is impossible
  ➢ We saw that even EQ+PO allocations may not exist

Open Problem
Does there exist a direct revelation, deterministic SP+EQ mechanism?
Randomized SP Mechanisms

• We want the mechanism *always* return an allocation satisfying a subset of \{EQ,EF,PO\}, and be SP in expected utilities

• Recall: PO+EQ allocations may not exist
  ➢ Hence, we can only hope for SP+PO+EF or SP+EF+EQ
  ➢ The first is an open problem, but the second combination is achievable!

**Open Problem**

Does there exist a randomized SP mechanism which always returns a PO+EF allocation?
Randomized SP Mechanisms

• **Theorem [Mossel and Tamuz, 2010; Chen et al. 2013]**
  There is a randomized SP mechanism that *always* returns an EF+EQ allocation.

  ➢ Recall: In a perfect partition $B$, $v_i(B_k) = 1/n$ for all $i, k \in N$
  ➢ **Algorithm:** Compute a perfect partition and return allocation $A$ which randomly assigns the $n$ pieces to the $n$ agents

  ➢ **SP:** Regardless of what the agents report, agent $i$ receives each piece of the cake with probability $1/n$, and thus has expected value exactly $1/n$
  ➢ **EF:** Assuming agents report truthfully (due to SP), agent $i$ always receives a cake she values at $1/n$, and according to her, so do others.
Existential Summary

- **SP+PO+EF+EQ**: Impossibility
- **SP+PO+EQ**: Impossibility
- **SP+EF+EQ**: Possibility
- **PO+EF+EQ**: Impossibility

- **SP+PO**: Possibility
- **SP+EF**: Impossibility
- **SP+EQ**: Possibility
- **PO+EF**: Impossibility
- **PO+EQ**: Possibility
- **EF+EQ**: Possibility

\(\times\) = Impossibility
\(\checkmark\) = Possibility

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**SP+PO+EF**
- Det: Impossibility
- Rand: Impossibility

**SP+PO+EQ**
- Det: Impossibility
- Rand: Impossibility

**SP+EF+EQ**
- Det: Possibility
- Rand: Possibility

**PO+EF+EQ**
- Det: Possibility
- Rand: Possibility

**SP+PO**
- Det: Possibility
- Rand: Possibility

**SP+EF**
- Det: Impossibility
- Rand: Impossibility

**SP+EQ**
- Det: Possibility
- Rand: Possibility

**PO+EF**
- Det: Possibility
- Rand: Possibility

**PO+EQ**
- Det: Possibility
- Rand: Possibility

**EF+EQ**
- Det: Possibility
Special Cases
Piecewise Constant/Uniform Valuations

Piecewise constant density function

Piecewise uniform density function

Special case of piecewise constant
Possibilities

• **Theorem [Chen et al., 2013]**
  For piecewise uniform valuations, there exists a deterministic SP mechanism which returns an EF+PO allocation.
  - Recall that for general valuations, even deterministic SP+EF is impossible.

• **Theorem [Aziz and Ye, 2014]**
  For piecewise constant valuations, an s-CEEI (i.e. Nash-optimal) allocation can be computed in polynomial time.
  - Recall that this is EF (thus Prop) and PO.
  - But this is not SP.
EF in Robertson-Webb

- Theorem [Kurokawa et al., 2013]
  If an algorithm computes an envy-free allocation for \( n \) agents with piecewise uniform valuations with at most \( g(n) \) queries, then it can also compute an envy-free allocation for \( n \) agents with general valuations with at most \( g(n) \) queries.

- Let the same algorithm interact with general valuations \( v_1, \ldots, v_n \) via CUT and EVAL queries and return an allocation \( A \)

- The proof constructs piecewise uniform valuations \( u_1, \ldots, u_n \) which would have resulted in the same responses and \( u_i(A_j) = v_i(A_j) \) for all \( i, j \in N \)
PO in Robertson-Webb

• Non-wastefulness
  ➢ An allocation $A$ is called non-wasteful if no piece of the cake that is valued positively by at least one agent is assigned to an agent who has zero value for it
  ➢ PO implies non-wastefulness

• Theorem [Ianovski, 2012; Kurokawa et al., 2013]
  No finite protocol in the Robertson-Webb model can always produce a non-wasteful allocation, even for piecewise uniform valuations.

• This is the reason we did not provide query complexity results when discussing PO
Burnt Cake Division
Model

• Same as regular cake, except agents now have non-positive valuation for every piece of the cake
  
  \[ f_i(x) \leq 0, \forall x \in [0,1] \]
  
  Hence, \( v_i(X) \leq 0, \forall X \in \mathcal{A} \)

• Equitability and perfect partitions carry over from the goods case
  
  \[ \text{Simply use } -f_i \text{ and } -v_i \]
Dividing a Burnt Cake

• Theorem [Peterson and Su, 2009]
  For burnt cake division, there exists a finite (but unbounded) protocol for finding an envy-free allocation with \( n \) agents.
  ➢ Builds upon the Brams-Taylor protocol for dividing a good cake
  ➢ But certain operations require non-trivial transformations to the world of chores

Open Problem
Is there a bounded envy-free protocol for burnt cake division?
(Homogeneous) Divisible Goods
Model

- Agents: \( N = \{1, 2, \ldots, n\} \)
- Resource: Set of divisible goods \( M = \{g_1, g_2, \ldots, g_m\} \)
- Allocation \( A = (A_1, \ldots, A_n) \)
  \[ A_i = (A_{i,j})_{j \in [m]} \]
  \[ \forall i, j: A_{i,j} \in [0,1] \]
  \[ \forall j: \sum_i A_{i,j} \leq 1 \]

- Assume additive valuations \( v_i(A_i) = \sum_j A_{i,j} v_i(g_j) \)
- Special case of cake cutting (up to normalization)
**$n = 2$: Adjusted Winner Procedure**

[Brams and Taylor 1996]

- **Input:** Normalized valuation functions
- Order the goods by ratio $v_1(g)/v_2(g)$.

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$v_1(g)/v_2(g)$ high $\rightarrow$ $v_1(g)/v_2(g)$ low

- Divide the goods so that agent 1 receives goods $g_1, \ldots, g_{j-1}$, agent 2 receives goods $g_{j+1}, \ldots, g_m$ for some $j$, and $v_1(A_1) = v_2(A_2)$
  - $g_j$ is divided between the agents, if necessary
Theorem [Brams and Taylor 1996]:

- The adjusted winner procedure is envy-free (and therefore proportional), equitable and Pareto optimal

Breaks down for $n > 2$

- As in cake cutting, EF + EQ + PO is impossible, what about two of the three?
- EF+EQ: Divide each good equally among agents (“perfect partition”)
- EQ + PO: Impossible
- EF + PO: Can achieve with CEEI

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CEEI

• With a fixed set of items, the definition of s-CEEI (that we will now call just CEEI) becomes simpler.

• Equilibrium price $p_j > 0$ for each good $g_j$
  ➢ Assume for simplicity that $\forall j \exists i$ with $v_i(g_j) > 0$

• CE: If $A_{i,j} > 0$ then $\frac{v_i(g_j)}{p_j} \geq \frac{v_i(g_k)}{p_k}$ for all $k$

• EI: $\sum_j p_j A_{i,j} = 1$ for all $i$
Eisenberg-Gale convex program

• Can compute a CEEI allocation as the solution to the Eisenberg-Gale [1959] convex program:

\[
\begin{align*}
\max & \sum_{i \in N} \log u_i \quad \text{s.t.} \\
\forall i: & u_i \leq \sum_{g_j \in M} A_{i,j} v_i(g_j) \\
\forall j: & \sum_{i \in N} A_{i,j} \leq 1 \\
\forall i, j: & A_{i,j} \geq 0
\end{align*}
\]

• Theorem [Orlin 2010, Végh 2012]:
  ➢ The Eisenberg-Gale convex program can be solved in strongly polynomial time.
Strategyproofness

• CEEI solution is fair and efficient but not strategyproof.
  ➢ It is strategyproof in the large (SP-L) [Azevedo and Budish 2018] though

• Theorem [Han et al. 2011]:
  ➢ No strategyproof mechanism that always outputs a complete allocation can
    achieve better than a $1/m$ approximation to the optimal social welfare for large
    enough $n$.
    o Social welfare = $\sum_{i \in N} v_i(A_i)$

• Theorem [Cole et al. 2013]:
  ➢ There is a strategyproof partial allocation mechanism that provides every agent
    with a $1/e$ fraction of their CEEI utility.
  ➢ Allocation is envy-free but not proportional
SP + Prop + EF

• SP + Prop + EF is trivial! Just allocate everyone an equal fraction of each good.
  ➢ What if we also want PO?

• Theorem [Schummer 1996]:
  ➢ It is impossible to achieve SP + Prop + PO.
  ➢ SP + PO: Serial dictatorship.

• SP + Prop + EF can also be achieved non-trivially [Freeman et al. 2019]
  ➢ Additionally achieves strict SP: agents always achieve strictly higher utility by reporting their beliefs truthfully than by lying.
  ➢ Exploits a correspondence between fair division and wagering mechanisms [Lambert et al. 2008] to utilize proper scoring rules (e.g. Brier score)
Allocating Divisible Goods + Bads
Model

- Agents: $N = \{1, 2, \ldots, n\}$
- Resources: Set of divisible “items” $M = \{o_1, o_2, \ldots, o_m\}$
- Allocation $A = (A_1, \ldots, A_n)$
  - $A_i = (A_{i,j})_{j \in [m]}$
  - $\forall i, j: A_{i,j} \in [0,1]$
  - $\forall j: \sum_i A_{i,j} \leq 1$
- Assume additive valuations: $v_i(A_i) = \sum_j A_{i,j} v_i(o_j)$
  - However, $v_i(o_j)$ can be positive, zero, or negative
- We’ll refer to s-CEEI simply as CEEI in this case
Achieving EF+PO

• Theorem [Bogomolnaia et al. 2017]
  ➢ There always exists a CEEI allocation, which is envy-free and Pareto optimal.
  ➢ The CEEI solution is “welfarist”, i.e., the set of feasible utility profiles is enough to identify the set of CEEI utility profiles.
  ➢ The CEEI utility profile is given by the following:
    1. If it is possible to give a positive utility to each agent (who can receive a positive utility), then maximizing the Nash welfare gives the unique CEEI utility profile.
    2. Else, if the all-zero utility profile is feasible and Pareto optimal, then it is the unique CEEI utility profile.
    3. Else, there can be exponentially many CEEI utility profiles, which give non-positive utility to each agent.
  ➢ Their actual result is stronger and in a more general model
Not Covered

• Nash equilibria of cake-cutting
• Optimal cake-cutting
  ➢ Algorithms for maximizing social welfare subject to fairness constraints
• Number of cuts and moving knives protocols
  ➢ Possibility and impossibility results for $n - 1$ cuts
• Multidimensional cakes
• Randomized or strategyproof Robertson-Webb protocols
• Non-additive valuations
• ...
Indivisible Goods
Model

• Agents: \( N = \{1, 2, \ldots, n\} \)

• Resource: Set of indivisible goods \( M = \{g_1, g_2, \ldots, g_m\} \)

• Allocation \( A = (A_1, \ldots, A_n) \in \Pi_n(M') \) is a partition of \( M' \) for some \( M' \subseteq M \).

• Each agent \( i \) has a valuation \( v_i : 2^M \rightarrow \mathbb{R}_+ \)
  \( \triangleright v_i : 2^M \rightarrow \mathbb{R}_- \) in the case of \text{bads}, \( v_i : 2^M \rightarrow \mathbb{R} \) for both \text{goods and bads}
Valuation Functions

- **Additive**: $\forall X, Y$ with $X \cap Y = \emptyset$: $v_i(X \cup Y) = v_i(X) + v_i(Y)$
  - Equivalently: $v_i(X) = \sum_{g \in X} v_i(g)$
  - Value for a good independent of other goods received

- **Submodular**: $\forall X, Y$: $v_i(X \cup Y) + v_i(X \cap Y) \leq v_i(X) + v_i(Y)$
  - Equivalently: $\forall X, Y$ with $X \subseteq Y$: $v_i(X \cup \{g\}) - v_i(X) \geq v_i(Y \cup \{g\}) - v_i(Y)$

- **Subadditive**: $\forall X, Y$ with $X \cap Y = \emptyset$: $v_i(X \cup Y) \leq v_i(X) + v_i(Y)$

- Submodular and subadditive definitions capture the idea of diminishing returns.
Need new guarantees!
Envy-Freeness up to One Good
Envy-Freeness up to One Good (EF1)
[Lipton et al 2004, Budish 2011]

• An allocation is envy-free up to one good (EF1) if, for all agents $i, j$, there exists a good $g \in A_j$ for which

\[ v_i(A_i) \geq v_i(A_j \setminus \{g\}) \]

• “Agent $i$ may envy agent $j$, but the envy can be eliminated by removing a single good from $j$’s bundle.”

➢ Note: We don’t consider $A_j = \emptyset$ a violation of EF1.
Round Robin Algorithm

- Fix an ordering of the agents $\sigma$.
- In round $k \mod n$, agent $\sigma_k$ selects their most preferred remaining good.

**Theorem:** Round robin satisfies EF1.

Animation Credit: Ariel Procaccia
Algorithm for Achieving EF1

• Envy graph: Edge from $i$ to $j$ if $i$ envies $j$

• Greedy algorithm [Lipton et al. 2004]
  - One at a time, allocate a good to an agent that no one envies
  - While there is an envy cycle, rotate the bundles along the cycle.
    - Can prove this loop terminates in a polynomial number of steps

• Removing the most recently added good from an agent’s bundle removes envy towards them.
• Neither this algorithm nor round robin is Pareto optimal.
EF1 with Goods and Bads [Aziz et al. 2019]

• An allocation is envy-free up to one item (EF1) if, for all agents $i, j$, there exists an item $o \in A_i \cup A_j$ for which

$$v_i(A_i \setminus \{o\}) \geq v_i(A_j \setminus \{o\})$$

• Round robin fails EF1

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Double Round Robin

• Let $O^{-} = \{o \in O: \forall i \in N, v_i(o) \leq 0\}$ denote all unanimous 
bads and $O^{+} = \{o \in O: \exists i \in N, v_i(o) > 0\}$ denote all objects that are a good 
for some agent.
  ➢ Suppose that $|O^{-}| = an$ for some $a \in \mathbb{N}$. If not, add dummy 
bads with $v_i(o) = 0$ for all $i \in N$.

• Double round robin:
  ➢ Phase 1: $O^{-}$ is allocated by round robin in order $(1, 2, \ldots, n - 1, n)$
  ➢ Phase 2: $O^{+}$ is allocated by round robin in order $(n, n - 1, \ldots, 2, 1)$
  ➢ Agents can choose to skip their turn in phase 2
Double Round Robin

• Theorem [Aziz et al. 2019]:
  - The double round robin algorithm outputs an allocation that is EF1 for combinations of goods and bads in polynomial time.
  - Proof idea: Let $i < j$. Agent $i$ can envy $j$ up to one item in phase 1 (but not vice versa), and agent $j$ can envy $i$ up to one item in phase 2 (but not vice versa).

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Maximum Nash Welfare

- **Maximum Nash Welfare (MNW):** Select the allocation that maximizes the geometric mean of agent utilities (more on this later).

\[ A = \arg \max \left( \prod_i v_i(A_i) \right)^{1/n} \]

➢ This is just Nash-optimality from earlier

- What if \( \prod_i v_i(A_i) = 0 \) for all allocations?
  ➢ Find an allocation that maximizes \(|\{v_i(A_i) > 0\}|\), and subject to that maximizes

\[ \left( \prod_{i:v_i(A_i)>0} v_i(A_i) \right)^{1/n} \]
EF1 + PO

• Theorem [Caragiannis et al. 2016]:
  ➢ The MNW allocation satisfies EF1 and PO.
  ➢ PO: A Pareto-improving allocation would have higher geometric mean of utilities for agents with non-zero utility or more agents with non-zero utility.
  ➢ EF1: Let $g^*_i = \arg \max_{g \in A_i} v_i(g)$. Not-too-hard proof shows $v_j(A_j) \geq v_j(A_i \setminus g^*_i)$ for all $j$.

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Computing EF1 + PO

• The MNW allocation is strongly NP-hard to compute (reduction from X3C).
  ➢ Actually, it’s APX-hard [Lee 2017].

• Special case: Binary valuations
  ➢ MNW allocation can be computed in polynomial time [Darmann and Schauer 2015, Barman et al. 2018].
  ➢ However, round robin already guarantees EF1 + PO in this setting.
Computing EF1 + PO

• Theorem [Barman et al. 2018]:
  ➢ There exists a pseudo-polynomial time algorithm for computing an allocation satisfying EF1 + PO
  ➢ Algorithm uses local search (sequence of item swaps and price rises) to compute an integral competitive equilibrium that is price envy-free up to one good.
  ➢ Price envy-free up to one good: \( \forall i, k, \exists j: p(A_i) \geq p(A_k \setminus \{g_j\}) \)
  ➢ Need different entitlements because CEEI might not exist with indivisibilities
    o Two agents, one item...
Computing EF1 + PO

Open Problem:
Complexity of computing an EF1 + PO allocation

Open Problem:
Does there always exist an EF1 + PO allocation for submodular valuation functions?
EF1 + PO for Bads

• Theorem [Aziz et al. 2019]:
  ➢ When items can be either goods or bads and $n = 2$, an EF1 + PO allocation always exists and can be found in polynomial time

Open Problem:
Does an EF1 + PO allocation always exists for bads?
Proportionality up to One
Good
An allocation is **proportional up to one good (Prop1)** if, for every agent $i$, there exists a good $g$ for which

$$v_i(A_i \cup \{g\}) \geq \frac{v_i(M)}{n}$$

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$v_1(A_1 \cup \{g_2\}) = 4 \geq \frac{7}{2} = \frac{v_i(M)}{n}$
Prop1 + PO

• Any algorithm that satisfies EF1 + PO is also Prop1 + PO.
  ➢ MNW
  ➢ Barman et al. [2018] algorithm

• Theorem [Barman and Krishnamurthy 2019]:
  ➢ An allocation satisfying Prop1 + PO can be computed in strongly polynomial time.

• Allocation is a careful rounding of the fractional CEEI allocation.
  ➢ In contrast, there exist instances in which no rounding of the fractional CEEI allocation will give EF1 [Caragiannis et al., 2016].
Envy-Freeness up to the Least Valued Good
Envy-Freeness up to the Least Valued Good
[Caragiannis et al. 2016]

An allocation is envy-free up to the least valued good (EFX) if, for all agents $i, j$, and every $g \in A_j$ with $v_i(g) > 0$,

$$v_i(A_i) \geq v_i(A_j \setminus \{g\}).$$
Leximin Allocation

• Leximin allocation:
  - First, maximize the minimum utility any agent receives. Subject to this, maximize the second-minimum utility. Then the third-minimum utility, etc.

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Satisfying EFX

• Theorem [Plaut and Roughgarden, 2018]:
  ➢ The Leximin allocation satisfies EFX + PO for agents with (general) identical valuations.

• Theorem [Plaut and Roughgarden, 2018]:
  ➢ The Leximin allocation satisfies EFX + PO for two agents with (normalized) additive valuations.

Open Problem:
Does there always exist a complete allocation satisfying EFX?
Satisfying EFX

• What about partial allocations satisfying EFX?
  ➢ Easy! We can just throw all goods away and take the empty allocation.

• Theorem [Caragiannis et al. 2019]:
  ➢ There exists a partial allocation that satisfies EFX and achieves a 2-approximation to the optimal Nash welfare.
  ➢ No (complete or partial) EFX allocation can achieve a better approximation.
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<tr>
<td>Prop1</td>
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Maximin Share
Maximin Share [Budish 2011]

• “If I partition the goods into \( n \) bundles and receive an adversarially chosen bundle, how much utility can I guarantee myself?”

• Define \( MMS_i^k(S) = \max_{(P_1, \ldots, P_k) \in \Pi_k(S)} \min_{1 \leq j \leq k} v_i(P_j) \)

• MMS allocation: One for which \( v_i(A_i) \geq MMS_i^n(M) \)

• Note that \( MMS_i^n(M) \leq \frac{v_i(M)}{n} \), so Proportionality implies MMS
Maximin Share [Budish 2011]

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$$MMS_1^n(M) = \min(3, 3, 3) = 3$$

$$MMS_2^n(M) = \min(10, 5, 5) = 5$$

$$MMS_3^n(M) = \min(4, 5, 5) = 4$$
Achieving Maximin Allocations

• Theorem [Procaccia and Wang 2014]:
  ➢ There exist instances for which no allocation satisfies MMS.

• Instead, consider approximations.
  ➢ c-MMS: allocation for which $v_i(A_i) \geq c \cdot MMS_i^n(M)$
  ➢ Guarantee $v_i(A_i) \geq MMS_i^k(M)$ for some $k > n$

• Theorem [Budish 2011]:
  ➢ There always exists an allocation that satisfies $v_i(A_i) \geq MMS_i^{(n+1)}(M)$ for every agent $i$. 
c-MMS Allocations

• Theorem [Procaccia and Wang 2014]:
  ➢ A (2/3)-MMS allocation always exists.

• Theorem [Amanatidis et al. 2017]:
  ➢ A (2/3-\(\epsilon\))-MMS allocation can be computed in polynomial time.

• Theorem [Ghodsi et al. 2018]:
  ➢ A (3/4)-MMS allocation always exists and a (3/4-\(\epsilon\))-MMS allocation can be computed in polynomial time.
c-MMS Allocations

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<th>Subadditive</th>
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<td>$\frac{1}{10} \log m$</td>
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<tr>
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<td>$\frac{3}{4} - \epsilon$</td>
<td>$\frac{1}{3}$</td>
<td>-</td>
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<tr>
<td>Upper bound</td>
<td>$1 - \frac{1}{n^{n+1}}$</td>
<td>$\frac{3}{4}$</td>
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Open Problem:
Close the gaps!

[Ghodsi et al. 2018]
c-MMS Allocations for Bads

• Theorem [Aziz et al. 2017]:
  ➢ A 2-MMS allocation always exists and can be computed in polynomial time when dividing bads.

• Theorem [Barman and Krishnamurthy 2017]:
  ➢ A (4/3)-MMS allocation always exists and can be computed in polynomial time when dividing bads.
Groupwise MMS [Barman et al. 2018]

- Idea: $MMS_i^k$ should be guaranteed for all groups $J$ of agents of size $k$ and set of goods $\cup_{i \in J} A_i$

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- $\nu_3(A_3) \geq MMS_3^2(M)$ but $\nu_3(A_3) < MMS_3^2(A_1 \cup A_3)$
Groupwise MMS [Barman et al. 2018]

• Allocation $A$ satisfies Groupwise Maximin Share (GMMS) if,

\[ \forall i: v_i(A_i) \geq \max_{J \subseteq N} MMS_i^{|J|}(\bigcup_{j \in J} A_j) \]

• Theorem [Barman et al. 2018]:
  - When valuations are additive, a 0.5-GMMS allocation exists and can be found in polynomial time.
  - Algorithm: Select an agent who is not envied by any other agent, and allocate her her most preferred unallocated good.
  - Small refinement of EF1 algorithm from earlier
(Relaxed) Equitability
Equitability

• Recall equitability:

\[ \forall i, j \in N: v_i(A_i) \geq v_j(A_j) \]

• We can relax it in the same way we did for envy-freeness [Gourves et al. 2014, Freeman et al. 2019].

• Equitability up to one good (EQ1):

\[ \forall i, j \in N, \exists g \in A_j: v_i(A_i) \geq v_j(A_j \setminus \{g\}) \]

• Equitability up to any good (EQX):

\[ \forall i, j \in N, \forall g \in A_j: v_i(A_i) \geq v_j(A_j \setminus \{g\}) \]
Algorithm for Achieving EQX

• Greedy Algorithm [Gourves et al. 2014]:
  - Allocate to the lowest-utility agent the unallocated good that she values the most.

• Almost the same as EF1 algorithm, but achieves EQX!
  - Compare to EFX, existence still unknown
EQ1/EQX + PO

• Theorem [Freeman et al. 2019]:
  - An allocation satisfying EQ1 and PO may not exist.
  - Compare to EF1 + PO always exists

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• Theorem [Freeman et al. 2019]:
  - When valuations are strictly positive, the Leximin allocation is EQX + PO
Group Fairness
Beyond Individual Fairness

Envy-Free up to One Good (EF1)
Group Fairness

• An allocation $A$ is group fair if for every non-empty $S, T \subseteq N$ and every partition $(B_i)_{i \in S}$ of $\cup_{j \in T} A_j$, $(\frac{|S|}{|T|}) \cdot (v_i(B_i))_{i \in S}$ does not Pareto dominate $(v_i(A_i))_{i \in S}$

• “It should not be possible to redistribute the goods allocated to group $T$ amongst group $S$ in such a way that every member of group $S$ is (weakly, with at least one strictly) better off, with utilities adjusted for group sizes”

• Group Fairness $\Rightarrow$ EF + PO
Group Fairness Relaxations

• Group Fairness up to One Good, After (GF1A) [Conitzer et al. 2019]
  ➢ “It should not be possible to redistribute the goods allocated to group T amongst group S in such a way that every member of group S is (weakly, with at least one strictly) better off, even when one good is removed from each agent in S, with utilities adjusted for group sizes”
Group Fairness Relaxations

- **Group Fairness up to One Good, Before (GF1B) [Conitzer et al. 2019]**
  - “It should not be possible to redistribute the goods allocated to group T, with one good per agent in T removed, amongst group S in such a way that every member of group S is (weakly, with at least one strictly) better off, with utilities adjusted for group sizes”

![Diagram showing partition and allocation]
Group Fairness Relaxations

• **Group Fairness up to One Good, Before (GF1B)** [Conitzer et al. 2019]
  
  “It should not be possible to redistribute the goods allocated to group T, with one good per agent in T removed, amongst group S in such a way that every member of group S is (weakly, with at least one strictly) better off, with utilities adjusted for group sizes”

![Diagram showing Group Fairness up to One Good, Before (GF1B)]
Achieving GF1A/GF1B

• **Locally Nash-optimal allocation**: Product of utilities cannot be improved by moving a single good.
  \[
  \forall i, j, g \in A_j: v_j(g) > 0 \text{ and } v_i(A_i) \cdot v_j(A_j) \geq v_i(A_i + g) \cdot v_j(A_j - g)
  \]

• **Theorem [Conitzer et al. 2019]**:
  - Any locally Nash-optimal allocation satisfies GF1A and GF1B.
  - Can be computed in pseudo-polynomial time by local search
  - When valuations are identical, an allocation is locally Nash-optimal iff it is EFX/EQX.

Open Problem:
Can we compute a locally Nash-optimal allocation in polynomial time?
Known Groups

• When we want to provide guarantees for all subsets of agents, “up to one good per agent” guarantees are the best we can give.

Open Problem:
Can we give stronger guarantees when $S$ and $T$ are fixed in advance?
Nash Welfare Approximation
Nash Welfare Approximation

• We have seen that MNW satisfies several nice properties.
  ➢ GF1A/B ($\Rightarrow$ EF1) + PO
  ➢ Scale-free
  ➢ Natural fairness/efficiency tradeoff

• But NP-hard to optimize. Can we approximate?

• Theorem [Lee 2017]
  ➢ Computing an allocation that maximizes the geometric mean of agent utilities under additive valuation functions is APX-hard.
  ➢ Approximating to within a factor of 1.00008 is NP-hard.
Nash Welfare Approximation

• Theorem [Cole and Gkatzelis 2015, Cole et al 2017]:
  ➢ There exists a polynomial time algorithm that approximates the MNW objective to within a constant multiplicative factor of 2.

• Theorem [Barman et al. 2018]:
  ➢ There exists a polynomial time algorithm that approximates the MNW objective to within a constant multiplicative factor of 1.45.

Open Problem:
Close the gap between the 1.00008 lower bound and 1.45 upper bound.
Nash Welfare Approximation

• Approximate MNW solutions may not retain the nice properties of the exact solution.

• Theorem [Garg and McGlaughlin 2019]:
  ➢ There exists a polynomial time algorithm that approximates the MNW objective to within a constant multiplicative factor of 2 and achieves Prop1, (1/2n)-MMS and PO.

• And recall, there exists a partial allocation that satisfies EFX and is a 2-approximation to MNW objective [Caragiannis et al 2019].
Price of Fairness
Price of Fairness

• What effect does requiring a fairness property have on the social welfare?

• Price of Fairness [Bertsimas et al. 2011, Caragiannis et al. 2012]:
  ➢ The price of fairness of fairness property $P$ is defined as the ratio of the maximum possible social welfare and the maximum social welfare of an allocation that satisfies $P$.

• Strong Price of Fairness [Bei et al. 2019]:
  ➢ The strong price of fairness of fairness property $P$ is defined as the ratio of the maximum possible social welfare and the minimum social welfare of an allocation that satisfies $P$.

• Cf. Price of Stability and Price of Anarchy
Price of Fairness

• Theorem [Caragiannis et al. 2012]:
  ➢ The price of fairness for proportionality, envy-freeness and equitability are:

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<td>Equitability</td>
<td>$\infty$</td>
<td>$\Theta(n)$</td>
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• Caragiannis et al. also studied divisible items, and bads.
Price of Fairness

• Theorem [Bei et al. 2019]:
  ➢ Bounds on the (strong) price of fairness for indivisible goods

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Open Problem: Close the gap for EF1
Strategyproofness
Adding Strategyproofness

• None of the rules we have considered so far are strategyproof

• For divisible goods, structure of strategyproof mechanisms is fairly rich
  ➢ Impossibilities from the divisible realm carry over

• What about indivisible goods?

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Picking-Exchange Mechanisms
[Amanatidis et al. 2017]

• Picking Mechanism:
  ➢ Partition $M = N_1 \cup N_2$
  ➢ Agent 1 receives a subset of offers $O_1 \subseteq 2^{N_1}$. Let $S_1 = \arg\max_{S \in O_1} v_1(S)$.
  ➢ Agent 2 receives a subset of offers $O_2 \subseteq 2^{N_2}$. Let $S_2 = \arg\max_{S \in O_2} v_2(S)$.
  ➢ $A_1 = S_1 \cup (N_2 \setminus S_2)$ and $A_2 = S_2 \cup (N_1 \setminus S_1)$

• $N_1 = \{g_1, g_2, g_3, g_4\}, N_2 = \{g_5, g_6\}$

• $O_1 = \{\{g_1, g_2\}, \{g_2, g_3\}, \{g_4\}\}, O_2 = \{\{g_5\}, \{g_6\}\}$

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Picking-Exchange Mechanisms
[Amanatidis et al. 2017]

- **Exchange Mechanism:**
  - Partition $M = E_1 \cup E_2$
  - Set of exchange deals $D = \{(S_1, T_1), ..., (S_k, T_k)\}$, where each $(S, T) \subseteq (E_1, E_2)$
  - Agent $i$ receives allocation $E_i$ by default, with exchanges performed if they are mutually beneficial

- $E_1 = \{g_1, g_2, g_3\}, E_2 = \{g_4, g_5\}$
- $D = \{\{g_2, g_3\}, \{g_4\}\}$

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Picking-Exchange Mechanisms

[Amanatidis et al. 2017]

• **Picking-Exchange Mechanism:** Run a picking mechanism on $N_1 \cup N_2 \subseteq M$ and an exchange mechanism on $E_1 \cup E_2 \subseteq M$, where $N_1 \cup N_2 \cup E_1 \cup E_2 = M$ and $N_1, N_2, E_1, E_2$ are pairwise disjoint.

  ➢ Up to tiebreaking technicalities...
Picking-Exchange Mechanisms
[Amanatidis et al. 2017]

• Theorem [Amanatidis et al. 2017]:
  ➢ For $n = 2$ an allocation mechanism that allocates all goods is strategyproof if and only if it is a picking-exchange mechanism

• Corollary [Amanatidis et al. 2017]:
  ➢ For $n = 2$, any strategyproof mechanism that allocates all goods does not achieve any positive approximation of the minimum envy or best proportionality guarantee.
  ➢ For $n = 2$ and $m \geq 5$, no strategyproof mechanism can allocate all items and satisfy EF1.
  ➢ For $n = 2$, no strategyproof mechanism guarantees better than $\frac{1}{m/2}$-MMS.
    o This is a tight bound [Amanatidis et al. 2016]
Open Problem:
What is the structure of strategyproof mechanisms for $n = 2$ when not all goods have to be allocated?

Open Problem:
What is the structure of strategyproof mechanisms for $n > 2$?
What’s Not Covered

• Envy-freeness up to one less-preferred item (EFL) [Barman et al. 2018]
  ➢ Stronger than EF1 and guaranteed to exist
  ➢ Existence of EFL + PO allocations is an open question

• Various constraints and additional features
  ➢ Agent social network structure
  ➢ Connectivity constraints when items lie on a graph

• Asymptotic results

• ...
Ordinal Preferences
Ordinal Preferences

• Instead of valuation functions, take in preference orderings $\succeq_i$ over items
  ➢ E.g. $g_2 \succeq_i g_3 \succeq_i g_1 \succeq_i g_4$

• Agents are assigned fractions of each item
  ➢ $A = (A_{i,j})_{i \in [n], j \in [m]}
  ➢ Can be interpreted as lotteries over integral allocations
Ordinal Preferences

• Partial preferences over bundles defined via stochastic dominance extension

\[ A \succeq_{SD}^i B \quad \text{iff} \quad \forall k: \sum_{j \succeq_{ik}} A_{i,j} \geq \sum_{j \succeq_{ik}} B_{i,j} \]

• Many other extensions possible
  ➢ Upper/downward lexicographic [Cho 2012]
  ➢ Pairwise comparison [Aziz et al. 2014]
  ➢ Bilinear dominance [Aziz et al. 2014]

• Can also elicit ordinal information over subsets directly [Bouveret et al. 2010]
Two Mechanisms

• **Random Priority**
  - Select a random ordering of the agents. Agents select their favorite \( m/n \) goods in order.

• **Probabilistic Serial [Bogomolnaia and Moulin 2001]**
  - Agents “eat” at a constant (equal) rate. At any time, agents eat their most preferred good that is not completely consumed.

\[
\succeq_1: g_1 \succeq_1 g_2 \succeq_1 g_3 \succeq_1 g_4
\]

**Random Priority**

\[
\begin{array}{cccc}
| \text{Agent} | g_1 & g_2 & g_3 & g_4 \\
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\end{array}
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\[
\succeq_2: g_2 \succeq_2 g_3 \succeq_2 g_1 \succeq_2 g_4
\]

**Probabilistic Serial**

\[
\begin{array}{cccc}
| \text{Agent} | g_1 & g_2 & g_3 & g_4 \\
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\]
SD-efficiency

• SD-efficiency: There should not exist an alternative allocation that all agents weakly prefer and some agent strictly prefers.

• Theorem [Bogomolnaia and Moulin 2001]:
  ➢ Probabilistic Serial satisfies SD-efficiency

• Random Priority is not SD-efficient
  ➢ $\succeq_1: g_1 \succeq_1 g_2 \succeq_1 g_3 \succeq_1 g_4$  
  ➢ $\succeq_2: g_2 \succeq_2 g_1 \succeq_2 g_4 \succeq_2 g_3$

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SD-strategyproofness

• SD-strategyproofness: No agent should be able to improve their allocation by misreporting their preferences.

• Theorem:
  ➢ Random Priority is SD-strategyproof.
  ➢ Probabilistic Serial is not SD-strategyproof

\[ \begin{align*}
  &\preceq_1: g_1 \succeq_1 g_2 \succeq_1 g_3 \succeq_1 g_4 \\
  &g_2 \succeq_1 g_1
\end{align*} \]

\[ \begin{align*}
  &\succeq_2: g_2 \succeq_2 g_3 \succeq_2 g_1 \succeq_2 g_4
\end{align*} \]
SD-Efficiency + SD-Strategyproofness

• Theorem [Bogomolnaia and Moulin 2001]:
  ➢ No mechanism satisfies SD-efficiency, SD-strategyproofness, and equal
treatment of equals

• We can get SD-efficiency + SD-envy-freeness
  ➢ SD-envy-freeness: \( \forall i, j: \sum_{j=1}^{m} A_{i,j}g_j \geq SD \sum_{j=1}^{m} B_{i,j}g_j \)
  ➢ Probabilistic Serial is SD-envyfree
Public Decisions
Public Decisions Model

• Set of agents $N$
• Set of issues $T$
• Each issue has associated set of alternatives $C^t = \{c^t_1, \ldots, c^t_{k_t}\}$
• Agents have utility functions $u^t_i: A^t \rightarrow \mathbb{R}_+$

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Item Allocation as a Special Case

- Define the set of issues $T = M = \{g_1, \ldots, g_m\}$
- Alternatives $C^t = N = \{a_1, \ldots, a_n\}$
- $u_i^t(a_j) = \begin{cases} v_i(g_t) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

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Fairness for Public Decisions

• Envy-freeness (and relaxations) not sensible in the general case
  ➢ Decisions are public, all agents receive the same outcome

• Proportionality is still sensible
  ➢ Each agent should receive their “dictator utility” multiplied by $1/n$

• Proportionality up to one issue (Prop1)
  ➢ Each agent would receive their proportional share if they were allowed to
    change the outcome on a single issue

• Theorem [Conitzer et al. 2017]:
  ➢ The MNW outcome satisfies Prop1 + PO in the public decisions setting

• Other fairness desiderata ((approximate) core, round robin share,...)
Allocation of Public Goods

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<thead>
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- Generalizes public decisions
- A set of public goods $\{g_1, \ldots, g_m\}$
  - Each good can give a positive utility to multiple agents simultaneously
- Constraints on which subsets of public goods are feasible
# Allocation of Public Goods

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</table>

- Public decision example:
  - Exactly one of $\{g_1, g_2, g_3\}$ and exactly one of $\{g_4, g_5, g_6\}$ must be chosen
  - Partition matroid constraint
Fairness Guarantees

• \((\delta, \alpha)\)-Core
  
  An allocation of public goods \(C\) is in \((\delta, \alpha)\)-core if for every subset of agents \(S \subseteq N\), there is no feasible allocation of public goods \(C'\) such that
  
  \[
  \frac{|S|}{n} \cdot u_i(C') \geq u_i(C)
  \]
  
  for all \(i \in S\), and at least one inequality is strict.

• Valuations are normalized so that \(\max_j u_i(g_j) = 1\)

• Core (i.e. \((0,0)\)-core) generalizes proportionality
  
  \((0,1)\)-core generalizes a guarantee very similar to Prop1
Fair Allocation of Public Goods

• Matroid constraints
  - Public goods are ground set elements
  - Feasible allocations are basis of a matroid
  - Generalizes public decisions (thus goods allocation) and multiwinner voting

• Theorem [Fain et al. 2018]
  - For matroid constraints, a (0,2)-core allocation exists, and for constant $\epsilon > 0$, a (0,2 + $\epsilon$)-core allocation can be computed in polynomial time.
  - Algorithm: Maximize smooth Nash welfare $\prod_{i \in N} (1 + u_i(C))$
  - For $\epsilon > 0$, (0,1 − $\epsilon$)-core allocations may not exist.

Open Problem: Does there always exist a (0,1)-core allocation?
Fair Allocation of Public Goods

- **Theorem [Fain et al. 2018]**
  - For “matching constraints” and constant $\delta \in (0,1]$, a $(\delta, 8 + 6/\delta)$-core allocation can be computed in polynomial time.
  - Algorithm: Maximize a slightly different smooth NW $\prod_{i \in N} \left(1 + 4/\delta + u_i(C)\right)$
  - For $\delta > 0$ and $\alpha < 1$, a $(\delta, \alpha)$-core allocation may not exist.
  - Open problem: Does there always exist a $(0,1)$-core allocation?

- A slightly worse guarantee with logarithmically large $\alpha$ in case of “packing constraints”
References


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